SOPHOMORE'S DREAM FUNCTION

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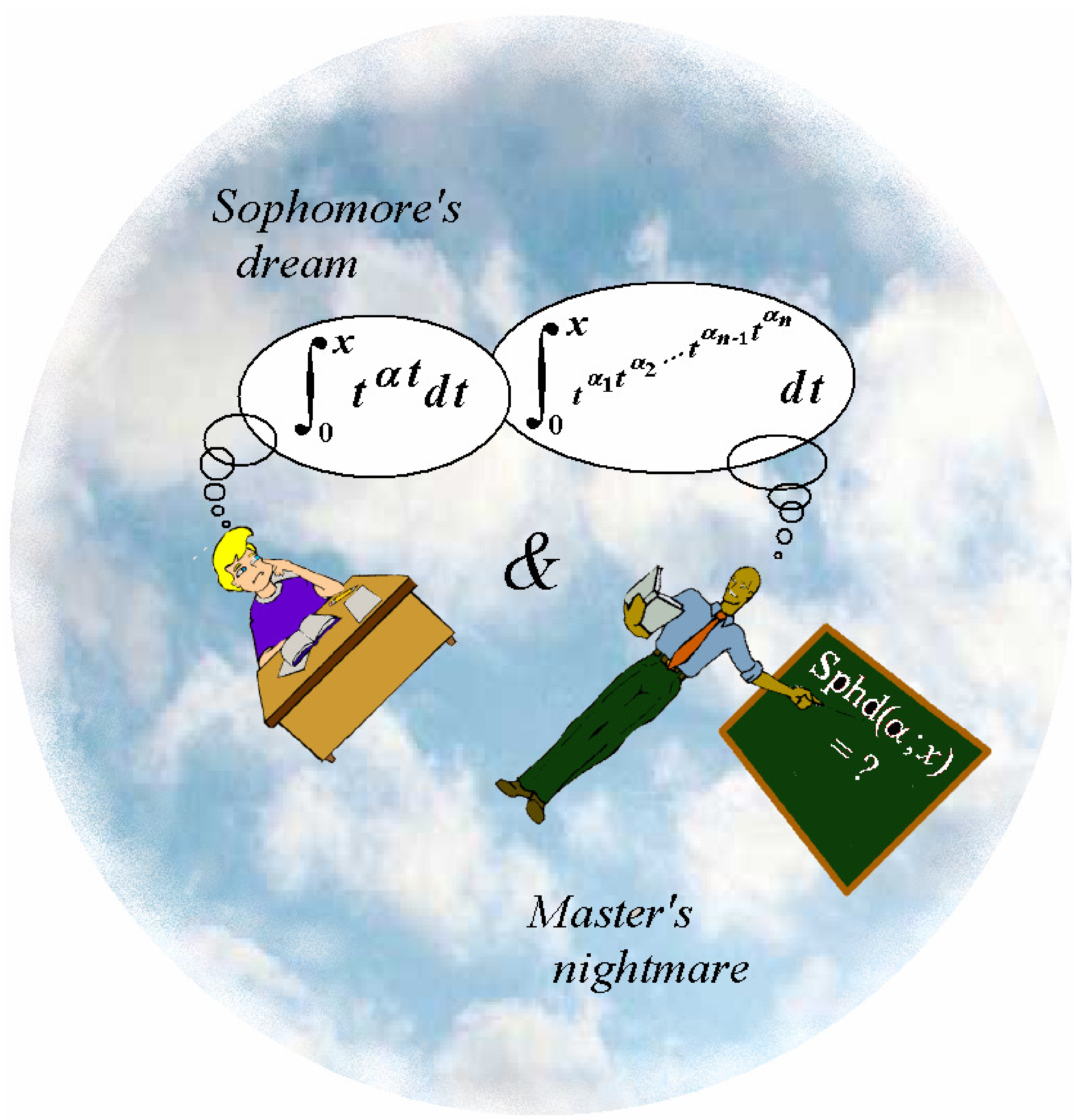

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Preamble : A schoolboy joke

Students could have heard about the "sophomore's dream" which refers to an old mathematical problem : Almost an hackneyed subject, which still appears from time to time on the WEB on mathematical forums and chats. For example, let us chuckle at a funny discussion (in interest of space only extracts are reported from [1]) :

integral of x^x. exists? Apr5-06, 08:59 AM

Is there a solution for $\int x^x dx$, I wonder...

Re: integral of x^x. exists? Jul2-10, 03:21 PM

This integral doesn't have a solution . Say will be great to know the solution of this integral
Function x^x
This function cannot be integrated. Most likely it's a nonelementary integral.
from http://www.numberempire.com/integralcalculator.php.

Re: integral of x^x. exists? Jul2-10, 03:25 PM

1. Please, do not do paleontological research here . This thread is 4 years old
2. x^x is an integrable function, because it is continuous

Re: integral of x^x. exists? Jul2-10, 03:57 PM

I am new here and can you give me this function ? I love it to see it .

Re: integral of x^x. exists? Jul2-10, 04:28 PM

Indeed. It is called F(x), and can be represented as follows:

$$F(x) = \int_0^x t^t dt$$

A number of other representations of it is possible, none of them in terms of a finite combination of elementary functions.

Re: integral of x^x. exists? Jul3-10, 07:21 AM

By the way, an integral of x^x is knows as a special function, namely the "Sophomore's dream" function.

Re: integral of x^x. exists? Jul3-10, 02:15 PM

Also, the "Sophomores Dream" usually refers to the result

$\int_0^1 \frac{1}{x^x} dx = \sum_{n=1}^{\infty} \frac{1}{n^n}$ also sometimes it is used to refer to

$\int_0^1 x^x dx = -\sum_{n=1}^{\infty} (-n)^{-n}$. It never refers to a function.

Re: integral of x^x. exists? Jul3-10, 03:25 PM

Well, if not, why not the "Masters dream function" ?

It's hard to believe that nobody ever had the brilliant idea to define a special function of this kind :

$$\mathrm{Sphd}(x)=\int_0^x t^t\,dt$$: namely, the " **S**o**ph**omore's **d**ream " function.

Probably was this done in the good old days, when the special functions were in fashion. Probably, someone granted this magnificent function with a colorful but old-fashion name. Probably all this long-forgotten stuff is peacefully sleeping among outdated manuscripts along other special functions bearing a similar unlucky fate.

What a pity ! If the Sophomore's dream function were still alive, when a student, eager for knowledge, wonder in this manner :
" Is there a primitive of x^x ? I'd love it to see it ! "
It should be so easy to answer :
" Indeed, it's Sphd(x), the Sophomore's dream function."
But nowadays, young people are no longer of the kind they were one century ago :
" Sir, are you pulling my legs ? That's only a name given to the integral, nothing more ! "
That is, as we say "to be on the slippery slope", but there is a narrow escape !
"Well, young man, do you know a primitive of the function $1/x$? "
" Of course, I know. It's : $\ln(x)=\int_1^x\left(\frac{1}{t}\right)dt$, namely a logarithm".
" Ha-ha! Are you pulling my legs young man ! That's only a name given to the integral … Nothing more … are you sure ? ! "

Comparison of the status of ln(x) and Sphd(x) may become controversial anyway. However this isn't the point here, we'll let student and teacher pursue in private meeting their pleasant discussion.

Respective status of functions in general and special functions, and among them elementary functions, are brought up in another paper written for the general public, entitled : "*Safari au pays des functions spéciales*", accessible through the link [2].

More seriously, what has to be done in order to make Sphd(x) a referenced and useful special function ? (of course, not a usual function like ln(x) : let's neither be fooled, nor daydream, even for a dream function).
A special function, as defined in [3-4], has to acquire a background of property, descriptions, formulas and derivations as extended as possible. So, it will be preferable to simply refer to a particular part of the background, instead of searching and redoing a development by ourselves. Before becoming a referenced special function, its name has to be spread in the literature in order to become familiar. More importantly, the function should be useful in a branch of mathematics or physics.

Generally, a function has a long way to go before acquiring the honorific status of special function. Probably a long way too for the Sophomore's dream function. Indeed, it will depend on many contributors. My own contribution will here appear so modest in the light of all that still remains to de done.

1. NOTATIONS

The denominations of sections and numbering of equations follow the rules established in "An Atlas of Functions" [7]

The name "sophomore's dream", introduced in [5], refers to particular cases I_1 and I_2 in terms of infinite sums :

1 : 1 $$I_1 = \int_0^1 x^x dx = -\sum_{n=1}^{\infty} \frac{(-1)^n}{n^n} \quad ; \quad I_2 = \int_0^1 x^{-x} dx = \sum_{n=1}^{\infty} \frac{1}{n^n}$$

The notations I_1 and I_2 , from [5-6] are not specific and may be confused with many others in the literature. Moreover, they don't refer to functions, but to constants.
That is why a more general notation : "Sphd" is introduced

1 : 2 $$\mathrm{Sphd}(\alpha\,;x) = \int_0^x t^{\alpha t}\,dt$$

so that $I_1 = \mathrm{Sphd}(1;1)$ and $I_2 = \mathrm{Sphd}(-1;1)$ are particular values of $\mathrm{Sphd}(\alpha\,;x)$.

Of course, presently Sphd isn't recognized officially and might be eventually replaced by a more convenient notation. The most important point is not here.

2. BEHAVIOR

Figure 1 maps $\mathrm{Sphd}(\alpha;\,x)$ in $x \geq 0$. The limit value $\mathrm{Sphd}(\alpha;\,0) = 0$ is stated in Section 7. When extending to negative x, $\mathrm{Sphd}(\alpha\,;x)$ goes in the complex domain, which will be considered only succinctly in Section 11.

As x increases, $\mathrm{Sphd}(\alpha \geq 0\,;x)$ increases more and more sharply while $\mathrm{Sphd}(\alpha < 0\,;x)$ reaches a ceiling. Behaviors for large magnitude of the argument will be derived in Section 9.

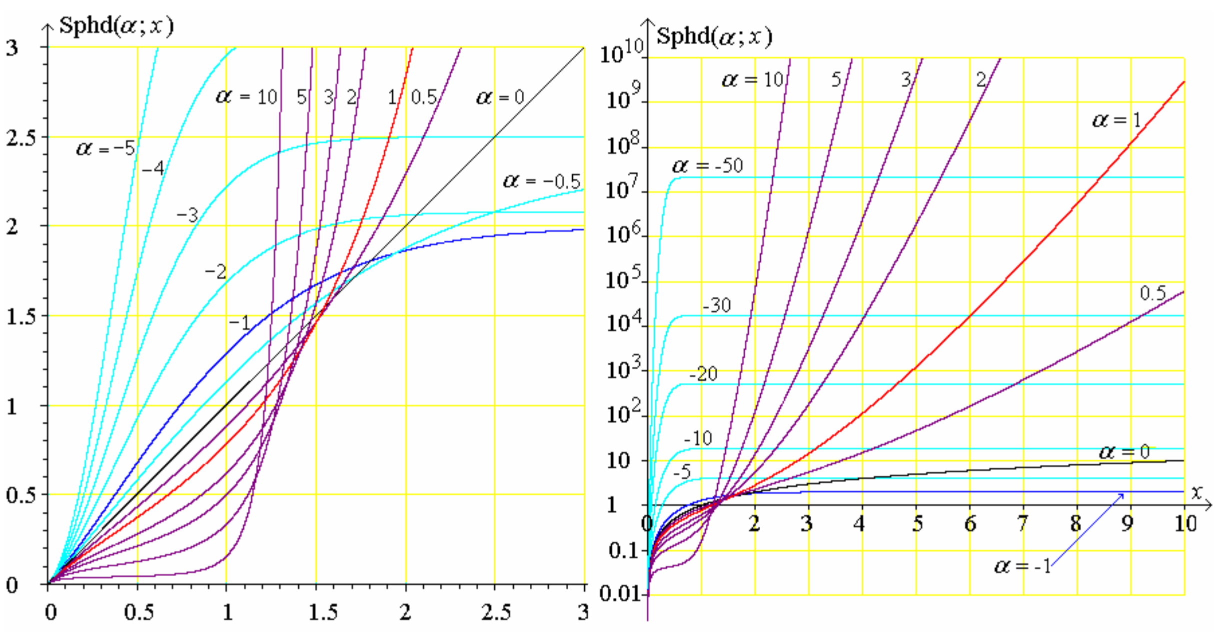


Figure 1.

3. DEFINITIONS

Variants for the integral definition are easily derived by some obvious transformations :

3 : 1 $$\mathrm{Sphd}(\alpha\,;x)=\int_0^x t^{\alpha t}\,dt=\int_0^x \left(t^t\right)^{\alpha}\,dt=\int_0^x e^{\alpha t\ln(t)}dt$$

3 : 2 $$\mathrm{Sphd}(\alpha\,;x)=\int_{-\infty}^{\ln(x)} e^{(\alpha e^{\theta}+1)\theta}\,d\theta \quad\rightarrow\quad \int_{-\infty}^{X} e^{(\alpha e^{\theta}+1)\theta}\,d\theta=\mathrm{Sphd}(\alpha\,;e^{X})$$

3 : 3 $$\mathrm{Sphd}\left(\alpha\,;x\right)=\int_{(1/x)}^{\infty} \tau^{-(2+(\alpha/\tau))}d\tau \quad\rightarrow\quad \int_{X}^{\infty} \tau^{-(2+(\alpha/\tau))}d\tau=\mathrm{Sphd}\left(\alpha\,;\tfrac{1}{X}\right)$$

3 : 4 $$\mathrm{Sphd}\left(\alpha\,;x\right)=\int_0^{\alpha x} \frac{T^T}{\alpha^{T+1}}dT \quad\rightarrow\quad \int_0^{X} \frac{T^T}{\alpha^{T+1}}dT=\mathrm{Sphd}\left(\alpha\,;\tfrac{X}{\alpha}\right)$$

4. SPECIAL CASES

When $\alpha=0$, reduction occurs : $\mathrm{Sphd}(0\,;x)=x$

5. INTRARELATIONSHIPS

Relationships in terms of geometric series :

5 : 1 $$\sum_{j=0}^{n-1}\mathrm{Sphd}(j\alpha\,;x)=\int_0^x \frac{1-t^{n\alpha t}}{1-t^{\alpha t}}dt$$

5 : 2 $$\sum_{j=0}^{n-1}(-1)^j\,\mathrm{Sphd}(j\alpha\,;x)=\int_0^x \frac{1+(-1)^n t^{n\alpha t}}{1+t^{\alpha t}}dt$$

5 : 3 $$\sum_{j=0}^{\infty}\mathrm{Sphd}(j\alpha\,;x)=\int_0^x \frac{1}{1-t^{\alpha t}}dt \qquad \text{in}\ \begin{cases} \alpha>0 \ \text{ and } \ 0\le x<1 \\ \text{or } \ \alpha<0 \ \text{ and } \ x>1 \end{cases}$$

5 : 4 $$\sum_{j=0}^{\infty}(-1)^j\,\mathrm{Sphd}(j\alpha\,;x)=\int_0^x \frac{1}{1+t^{\alpha t}}dt \qquad \text{in}\ \begin{cases} \alpha>0 \ \text{ and } \ 0\le x<1 \\ \text{or } \ \alpha<0 \ \text{ and } \ x>0 \end{cases}$$

6. EXPANSIONS

In terms of Incomplete Gamma function :

6 : 1 $$\mathrm{Sphd}\left(\alpha\,;x\right)=-\sum_{j=1}^{\infty}\frac{(-1)^j\alpha^{j-1}}{(j-1)!\,j^j}\Gamma\left(j\,;-j\ln(x)\right)$$

In terms of powers of x and $\ln(x)$:

6 : 2 $$\mathrm{Sphd}\left(\alpha\,;x\right)=-\sum_{j=1}^{\infty}\sum_{k=0}^{j-1}\frac{(-1)^{j-k}\alpha^{j-1}}{k!\,j^{j-k}}x^j\left(\ln(x)\right)^k$$

6 : 3 Series development by successive partial integrations

6 : 3 : 1
$$\int e^{\alpha x\ln(x)}dx=\frac{e^{\alpha x\ln(x)}}{\alpha\left(1+\ln(x)\right)}-\frac{1}{\alpha}\int e^{\alpha x\ln(x)}\frac{d}{dx}\left(\frac{1}{1+\ln(x)}\right)dx$$

6 : 3 : 2
$$F_0(x)=\frac{1}{1+\ln(x)}\ ;\quad F_1(x)=\frac{1}{1+\ln(x)}\frac{dF_0}{dx}\ ;\ldots;\quad F_n(x)=\frac{1}{1+\ln(x)}\frac{dF_{n-1}}{dx}$$

6 : 3 : 3
$$\int e^{\alpha x\ln(x)}dx=\sum_{n=0}^{N}\frac{(-1)^n}{\alpha^{n+1}}\frac{e^{\alpha x\ln(x)}}{1+\ln(x)}F_n(x)+\frac{(-1)^{N+1}}{\alpha^{N+1}}\int e^{\alpha x\ln(x)}\left(\frac{dF_N(x)}{dx}\right)dx$$

6 : 3 : 4
$$\begin{cases}\displaystyle\int_1^x e^{\alpha t\ln(t)}dt=\sum_{n=0}^{N}\frac{(-1)^n}{\alpha^{n+1}}\left(\frac{e^{\alpha x\ln(x)}}{1+\ln(x)}F_n(x)-F_n(1)\right)+I_{N+1}(x)\\ \displaystyle I_{N+1}(x)=\frac{(-1)^{N+1}}{\alpha^{N+1}}\int_1^x e^{\alpha t\ln(t)}\left(1+\ln(t)\right)F_{N+1}(t)\,dt\end{cases}$$

6 : 3 : 5
$$F_n(x)=\frac{(-1)^n}{x^n}\sum_{j=1}^{n}\frac{A_{n,j}}{\left(1+\ln(x)\right)^{n+j+1}}$$

6 : 3 : 6 Then, from 6 : 3 : 2 , the recurrence formula :

$$A_{n,j}=(n+j-1)A_{(n-1),(j-1)}+(n-1)A_{(n-1),j}\ ;\quad A_{n,1}=(n-1)!\ ;\quad A_{n,n}=\frac{(2n)!}{2^n\,n!}$$

n	$j = 1$	$j = 2$	$j = 3$	$j = 4$	$j = 5$
1	$A_{1,1} = 1$				
2	$A_{2,1} = 1$	$A_{2,2} = 3$			
3	$A_{3,1} = 2$	$A_{3,2} = 10$	$A_{3,3} = 15$		
4	$A_{4,1} = 6$	$A_{4,2} = 40$	$A_{4,3} = 105$	$A_{4,4} = 105$	
5	$A_{5,1} = 24$	$A_{5,2} = 196$	$A_{5,3} = 700$	$A_{5,4} = 1260$	$A_{5,5} = 945$

$\sum_{j=1}^{n} A_{n,j} = n^n$
1
4
27
256
3125

6 : 3 : 7 From 6 : 3 : 4 in substituting F_n from 6 : 3 : 5

$$\begin{cases}\displaystyle\int_1^x e^{\alpha t\ln(t)}dt=\frac{1}{\alpha}\left(\frac{e^{\alpha x\ln(x)}}{1+\ln(x)}-1\right)\\ \displaystyle\qquad-\frac{1}{\alpha}\sum_{n=1}^{N}\left(\frac{n}{\alpha}\right)^n\left(1-\sum_{j=1}^{n}\frac{A_{n,j}}{n^n}\frac{e^{\alpha x\ln(x)}}{x^n\left(1+\ln(x)\right)^{n+j+1}}\right)+I_{N+1}(x)\\ \displaystyle I_{N+1}(x)=\frac{1}{\alpha^{N+1}}\sum_{j=1}^{N+1}A_{N+1,j}\int_1^x\frac{e^{\alpha t\ln(t)}}{t^{N+1}\left(1+\ln(t)\right)^{N+j+1}}\,dt\end{cases}$$

6 : 4 Asymptotic expansion (αx large), from 6 : 3 : 7 :

$$\mathrm{Sphd}(\alpha,x)\ \sim\ \frac{e^{\alpha x\ln(x)}}{\alpha\left(1+\ln(x)\right)}\left(1+\sum_{n=1}^{n<<\alpha}\frac{1}{\alpha^n x^n}\sum_{j=1}^{n}\frac{A_{n,j}}{\left(1+\ln(x)\right)^{n+j}}\right)$$

The terms : $\mathrm{Sphd}(\alpha,1)$, $\left(-\frac{1}{\alpha}-\frac{1}{\alpha}\sum_{n=1}^{N}\left(\frac{n}{\alpha}\right)^n\right)$ and I_{N+1} with $N<<\alpha$ are negligible compared to the principal terms above.

6 : 5 Asymptotic expansion relatively to the parameter ($|\alpha|$ large) , from 6 : 3 : 7

$$\mathrm{Sphd}(\alpha,x)\ \sim\ \mathrm{Sphd}(\alpha,1)+\frac{1}{\alpha}\left(\frac{e^{\alpha x\ln(x)}}{1+\ln(x)}-1\right)$$

$$-\frac{1}{\alpha}\sum_{n=1}^{n<<|\alpha|}\left(\frac{n}{\alpha}\right)^n\left(1-\sum_{j=1}^{n}\frac{A_{n,j}}{n^n}\frac{e^{\alpha x\ln(x)}}{x^n\left(1+\ln(x)\right)^{n+j+1}}\right)$$

I_{N+1} is negligible compared to the principal terms.

7. PARTICULAR VALUES

(Numerical values in next section)

7 : 1 $\mathrm{Sphd}\left(\alpha\,;0\right)=0$ [From 6 : 1, where $\Gamma(j\,;\infty)=0$]

7 : 2 $\mathrm{Sphd}\left(\alpha\,;1\right)=-\sum_{j=1}^{\infty}\frac{(-1)^j}{j^j}\alpha^{j-1}$ [From 6 : 1, where $\Gamma(j\,;0)=(j\text{-}1)!$]

7 : 3 $I_1=\mathrm{Sphd}\left(1\,;1\right)=-\sum_{j=1}^{\infty}\frac{(-1)^j}{j^j}$ [From 7 : 2, particular case $\alpha=1$]

7 : 4 $I_2=\mathrm{Sphd}\left(-1\,;1\right)=\sum_{j=1}^{\infty}\frac{1}{j^j}$ [From 7 : 2, particular case $\alpha=-1$]

8. NUMERICAL VALUES

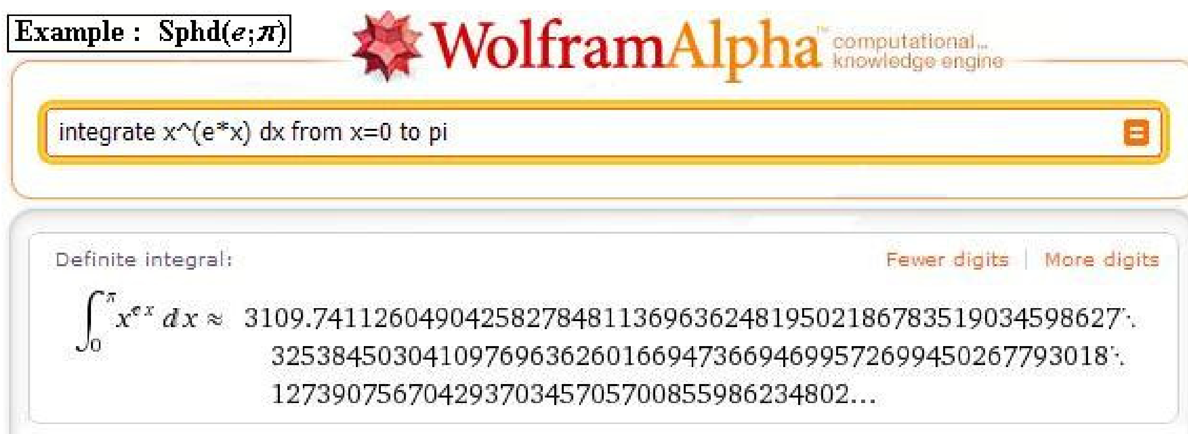
8 : 1 $\quad \mathrm{Sphd}(1;1) \approx 0.78343051071213\ldots$

8 : 2 $\quad \mathrm{Sphd}(-1;1) \approx 1.29128599706266\ldots$

8 : 3 $\quad \mathrm{Sphd}(-1;\infty) \approx 1.99545595750014\ldots$

Numerical values are commonly obtained by numerical integrations using various available mathematical packages for computers. Some are on free access on line. For example :

[http ://www.wolframalpha.com/input/?i=integrate+x%5E%28e*x%29+dx+from+x%3D0+to+pi](http://www.wolframalpha.com/input/?i=integrate+x%5E%28e*x%29+dx+from+x%3D0+to+pi)



Nevertheless, depending on the mathematical package used, the validity of all the posted digits can be questionable.

9. LIMITS AND APPROXIMATIONS

Notation : $\quad$ Relative Deviation $= \left| \dfrac{\text{Approximate value} - \text{Exact value}}{\text{Exact value}} \right|$

9 : 1 $\quad \mathrm{Sphd}(\alpha, x) \approx x + \dfrac{\alpha}{2} x^2 \ln(x) - \dfrac{\alpha}{4} x^2$ $\qquad$ Small x, from 6 : 2

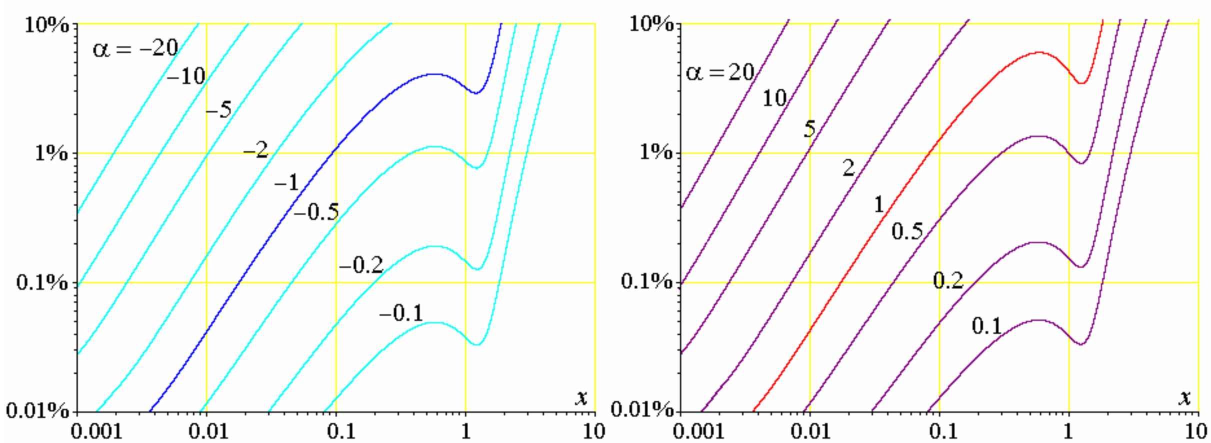


Figure 2 : Relative deviations for equation 9 : 1

9 : 2 $\quad \mathrm{Sphd}(\alpha,x) \sim \dfrac{e^{\alpha\, x \ln(x)}}{\alpha\,(1+\ln(x))}$ Equivalent in large x and α>0, from 6 : 4

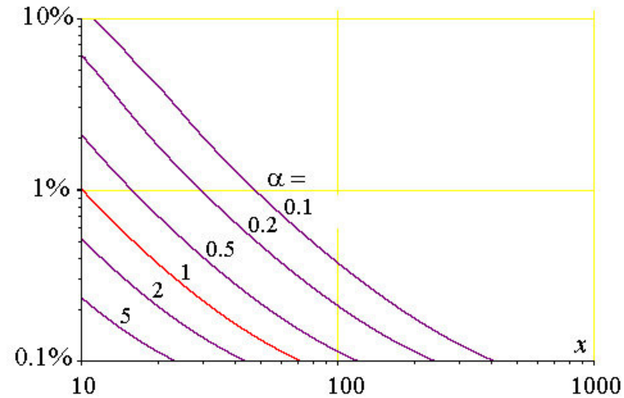


Figure 3 : Relative deviations for equation 9 : 2

9 : 3 $\quad \mathrm{Sphd}(\alpha,1) \sim \dfrac{1}{\alpha}\displaystyle\sum_{n=1}^{n<2|\alpha|}(-1)^{n+1}\left(\frac{\alpha}{n}\right)^{n}$ Equivalent in large |α|, from 7 : 2

Figure 4 maps the relative deviations. A much better precision can be achieved by increasing the number of terms of the series, for example with $n< 4\,|\alpha|$ instead of $n< 2\,|\alpha|$

9 : 4 $\quad \mathrm{Sphd}(\alpha,\infty) \sim \mathrm{Sphd}(\alpha,1)-\dfrac{1}{\alpha}-\dfrac{1}{\alpha}\displaystyle\sum_{n=1}^{n<|\alpha/2|}\left(\frac{n}{\alpha}\right)^{n}$ Equivalent in large |α|

case α<0 , from 6 : 4 and using approximate Sphd(α,1) from 9 : 3

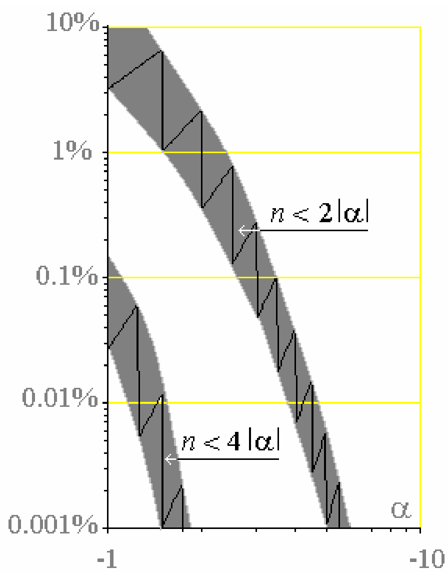


Figure 4
Range of relatives deviations
for equation 9 : 3

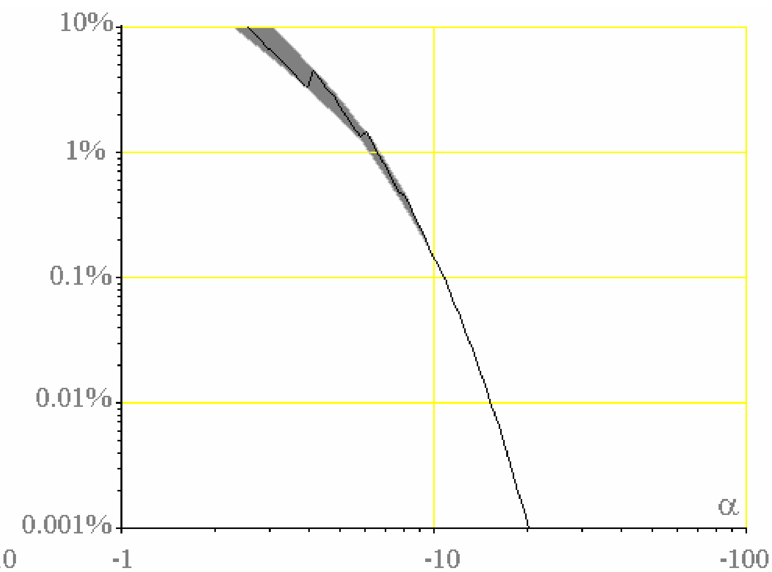


Figure 5
Range of relatives deviations
for equation 9 : 4

10. OPERATION OF THE CALCULUS

10 : 1 $\dfrac{d}{dx}\mathrm{Sphd}(\alpha;x)=x^{\alpha x}$

10 : 2 $\left(\dfrac{d}{dx}\mathrm{Sphd}(\alpha;x)\right)_{x=0}=1$ any α

10 : 3 $\left(\dfrac{d}{dx}\mathrm{Sphd}(\alpha;x)\right)_{x=1}=1$ any α

10 : 4 $\left(\dfrac{d}{dx}\mathrm{Sphd}(\alpha;x)\right)_{x=\infty}=\begin{cases}0 & \text{if } \alpha<0\\ 1 & \text{if } \alpha=0\\ \infty & \text{if } \alpha>0\end{cases}$

10 : 5 $\displaystyle\int \frac{1}{x}\mathrm{Sphd}(\alpha;x)dx=\left(1+\ln(x)\right)\mathrm{Sphd}(\alpha;x)-\frac{1}{\alpha}x^{\alpha x}$ $x>0$

10 : 6

$$\int_A^B \frac{1}{x}\mathrm{Sphd}(\alpha;x)dx=\left(1+\ln(B)\right)\mathrm{Sphd}(\alpha;B)-\frac{1}{\alpha}B^{\alpha B}-\left(1+\ln(A)\right)\mathrm{Sphd}(\alpha;A)+\frac{1}{\alpha}A^{\alpha A}$$ $A,\ B>0$

11. COMPLEX ARGUMENT

A general study in case of complex arguments isn't herein undertaken. Only the extension to negative arguments is reported :

11 : 1 $\displaystyle\mathrm{Sphd}(\alpha;x)=-\int_0^{-x}e^{-\alpha t\ln(t)}\cos(\pi\alpha t)dt+i\int_0^{-x}e^{-\alpha t\ln(t)}\sin(\pi\alpha t)dt$ $x<0$

12. GENERALIZATION

Various ways for generalization could be discussed. Often, a generalization draws to increase the number of parameters. This however leads to more difficulty in studying the generalized function and more and more particular cases to be considered.

So, the "Sophomore's dream function" is actually likely to become the "Master's dream function" (jokingly appearing in the preamble) or even the "Master's nightmare function" ! But introducing too many new names isn't such a good thing.

Even better, let's keep the name "Sophomore's dream" and only add an index to specify the number of parameters.

The integral of "nested exponentials", or "power towers", is proposed in order to generalize the Sphd(α;x) function. But there are several possibilities, such as :

$$(\alpha_1 x)^{\left((\alpha_2 x)^{\left((\alpha_3 x)^{\cdot^{\cdot^{\cdot^{(\alpha_n x)}}}}\right)}\right)} \quad \text{or} \quad x^{\alpha_1\left(x^{\alpha_2\left(\cdot^{\cdot^{\cdot^{x^{\alpha_{n-1}\left(x^{\alpha_n}\right)}}}}\right)}\right)}$$

The disadvantage of the first form lies in the case of negative parameter α which introduces complex values. After much hesitation, the preference is therefore given to the second form :

12 : 1
$$\mathrm{Sphd}_n(\alpha_1,\alpha_2,\alpha_3,\dots,\alpha_n;x)=\int_0^x \left(t^{\alpha_1}\right)^{\left(\left(t^{\alpha_2}\right)^{\left(\left(t^{\alpha_3}\right)^{\cdot^{\cdot^{\cdot^{\left(t^{\alpha_n}\right)}}}}\right)}\right)} dt$$

The typography of the tower in 12 : 1 looks different to the second form above, but both are equal in fact.

12 : 2 $\mathrm{Sphd}_2(\alpha\,,1\,;x)\equiv\mathrm{Sphd}(\alpha\,;x)$ is the relationship with the primary definition 3:1

12 : 3 $\mathrm{Sphd}_n(\,\alpha,\dots,\alpha\,;x)$ notation in case of n identical parameters.

In the case of the integrals of an infinite tower, a primary condition is the convergence of the tower on the range of integration. This raises the question of validity for the next notations :

12 : 4 $\mathrm{Sphd}_\infty(\alpha_1, \alpha_2, \alpha_3, \dots; x)$ is the formal notation in case of infinite tower

12 : 5 $\mathrm{Sphd}_\infty(1, \dots; x) = \int_0^x t^{t^{t^{\cdot^{\cdot^{\cdot}}}}} dt$ is the formal notation for the particular case of the integral of infinite tower with unit parameters.

Notations 12 : 4 and 12 : 5 are purely formal, since requiring the convergence of the tower power on the range of integration. Nevertheless, if the condition of convergence is satisfied, the difference of terms corresponding to the bounds of integration is significant.
For example :

12 : 6 $\mathrm{Sphd}_\infty(1, \dots; x_2) - \mathrm{Sphd}_\infty(1, \dots; x_1) = \int_{x_1}^{x_2} t^{t^{t^{\cdot^{\cdot^{\cdot}}}}} dt = -\int_{x_1}^{x_2} \frac{\mathrm{W}(-\ln(t))}{\ln(t)} dt$

where W is the Lambert W function [9] and $e^{-e} \leq x_1 \leq e^{1/e}$ and $e^{-e} \leq x_2 \leq e^{1/e}$

12 : 7 $\mathrm{Sphd}_\infty(-1, \dots; x_2) - \mathrm{Sphd}_\infty(-1, \dots; x_1) =$
$= \int_{x_1}^{x_2} (1/t)^{(1/t)^{(1/t)^{\cdot^{\cdot^{\cdot}}}}} dt = \int_{x_1}^{x_2} \frac{\mathrm{W}(\ln(t))}{\ln(t)} dt$
$e^{-e} \leq x_1 \leq e^{1/e}$ and $e^{-e} \leq x_2 \leq e^{1/e}$

12 : 8 $\mathrm{Sphd}_\infty(1, -1, \dots; x_2) - \mathrm{Sphd}_\infty(1, -1, \dots; x_1) =$
$= \int_{x_1}^{x_2} t^{(1/t)^{(1/t)^{\cdot^{\cdot^{\cdot}}}}} dt = \int_{x_1}^{x_2} \frac{\ln(t)}{\mathrm{W}(\ln(t))} dt$
$e^{-1/e} \leq x_1 \leq e^e$ and $e^{-1/e} \leq x_2 \leq e^e$

The conditions on x_1 and x_2 (Equations 12 : 6, 12 : 7 and 12 : 8) apply to the cases of infinite towers only.

13. COGNATE FUNCTIONS

Since various notations for the "power towers" or "iterated powers" [8] are used depending on the authors, their respective integrals are likely to be encountered in the literature with different symbols. This question of notations will be considered succinctly in the next section.

13 : 1 $\mathrm{Sphd}_n(1, \dots, 1; x) = \int_0^x {}^n t \, dt = \int_0^x \exp_t^n(1) \, dt = \int_0^x (t\uparrow)^n(1) \, dt$

This must not be confused with the integral of iterated exponential. $\int_0^x \exp_c^n(t) \, dt$

14. RELATED TOPICS.

The Sophomore's dream function, especially after generalization, is more or less related to some mathematical objects in hyper operator theories. The terminology is extensive, including a lot of terms such as : super exponentiation, hyper power, power tower, super root, super logarithm. Tetration is the fourth hyper operator :

1. Addition (The primary operation)

2. Multiplication $$x \times n = \underbrace{x + x + x + \ldots + x}_{n \text{ times}}$$

3. Exponentiation $$x^n = \underbrace{x \times x \times x \times \ldots \times x}_{n \text{ times}}$$

4. Tetration $$^{n}x = \underbrace{x^{x^{\cdot^{\cdot^{x}}}}}_{n \text{ times}}$$

Other notations for tetration : $^{n}x = x^{(4)}n = \text{hyper}_4(x,n) = (x\uparrow)^n(1) = \exp_x^n(1)$:

Notations for iterated exponential : $c^{c^{\cdot^{\cdot^{c^x}}}} = \exp_c^n(x) = \left(c\uparrow\right)^n(x)$ with n c's

More details, general information and a list of relevant references can be found in [10], but there is little material with regards to integrals of related functions.

REFERENCES

[1] http : //www.physicsforums.com/showthread.php?t=116575

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