On the Analytic Continuation of the Inverse of Iterated Exponentials to the Complex Plane

Andrew Robbins

2006-MM-DD

Abstract

Differentiability requirements for the inverse of iterated exponentials are shown to form a system of linear equations for the Taylor coefficients of this inverse. Any truncations of the associated matrix are shown to form approximations of this inverse. The existence, uniqueness and convergence of this inverse are then discussed. Some approximations of functions related to this inverse are also discussed.

1 Introduction

Using the notation $f^n(x) = f(f^{n-1}(x))$ used for iteration, and the notation $\exp_x(a) = x^a$ used for exponentials, a special case of iterated exponentials can be defined as follows.

Definition 1. Let ${}^y x = \exp_x^y(1),\ y \in \mathbb{Z}$ be the yth **tetration** of x.

The study of iterated exponentials began when Euler [1] showed the convergence of infinitely iterated exponentials (${}^\infty x$) over a closed interval. Maurer [3] later began using the notation ${}^n x$, which was recently popularized by Rucker [5]. This *left-superscript* notation has now become ubiquitous [1,4,?]. For brevity, Goodstein [2] coined the term *tetration* for iterated exponentials, and Rubtsov [4] uses the names *super-root*, and *super-logarithm* for the two inverses of tetration. Using Knuth's *up-arrow* notation [?], tetration can be written: ${}^y x = x \uparrow\uparrow y$. The terms *hyper-base* and *hyper-exponent* have been used to refer to x and y in $x \uparrow^n y$ respectively. Galidakis [?] uses the left-superscript notation in his work, as well as the notation ${}^y(x, a) = \exp_x^y(a)$ for iterated exponentials in general. Galidakis also uses the names *hyper-base* and *hyper-exponent* for x and y in ${}^y x$ respectively.

Definition 2. Let $\text{slog}_x(z) = y$ if $z = {}^y x$ be the **superlogarithm** to base-x of z.

This definition of the superlogarithm effectively creates a sparse domain for z (the range of tetration), since tetration is only defined for integer y. In order to extend the domains of tetration and the superlogarithm, a better understanding of their properties is required.

1.1 Axioms of Tetration and Superlogarithms

The axiom of tetration is:

$$\boxed{{}^{y}x = x^{({}^{(y-1)}x)}}, \tag{3}$$

which follows from (1). Given this property and a point (say, ${}^{0}x = 1$) tetration can be defined over integers as in (1), but it can also be used to find the axiom of superlogarithms.

Start with the definition of the superlogarithm, and replace y with $y+1$:

$$y = \operatorname{slog}_x({}^{y}x), \tag{4}$$

$$y + 1 = \operatorname{slog}_x({}^{(y+1)}x), \tag{5}$$

$$y = \operatorname{slog}_x({}^{(y+1)}x) - 1. \tag{6}$$

Then expand (6) using (3), and equate to (4):

$$y = \operatorname{slog}_x(x^{({}^{y}x)}) - 1, \tag{7}$$

$$\operatorname{slog}_x({}^{y}x) = \operatorname{slog}_x(x^{({}^{y}x)}) - 1. \tag{8}$$

Replacing ${}^{y}x$ with z, the result is the axiom of superlogarithms:

$$\boxed{\operatorname{slog}_x(z) = \operatorname{slog}_x(x^{z}) - 1}. \tag{9}$$

1.2 Derivatives of Tetration and Superlogarithms

Although we only know tetration for integer y, we can identify differential equations that are satisfied by tetration or superlogarithms, provided they are differentiable. Using the notations $\operatorname{tet}'_x(y) = \frac{\partial}{\partial y}({}^{y}x)$ and $\operatorname{slog}'_x(z) = \frac{\partial}{\partial z}\operatorname{slog}_x(z)$ for their derivatives, the derivative of (3) with respect to y will be of the form:

$$\operatorname{tet}'_x(y) = ({}^{y}x)\log(x)\operatorname{tet}'_x(y-1), \tag{10}$$

and the derivative of (9) with respect to z will be of the form:

$$\operatorname{slog}'_x(z) = (x^{z})\log(x)\operatorname{slog}'_x(x^{z}). \tag{11}$$

The inverse function derivative $(f^{-1})'(z) = 1/f'(f^{-1}(z))$ gives the forms:

$$\operatorname{tet}'_x(y) = \frac{1}{\operatorname{slog}'_x({}^{y}x)}, \tag{12}$$

$$\operatorname{slog}'_x(z) = \frac{1}{\operatorname{tet}'_x(\operatorname{slog}_x(z))}. \tag{13}$$

Evaluating at $x = \boldsymbol{e}$, $y = 0$ in (10, 12), and $x = \boldsymbol{e}$, $z = 0$ in (11, 13) gives:

$$\boxed{\operatorname{tet}'_{\boldsymbol{e}}(0) = \operatorname{tet}'_{\boldsymbol{e}}(-1) = \frac{1}{\operatorname{slog}'_{\boldsymbol{e}}(0)} = \frac{1}{\operatorname{slog}'_{\boldsymbol{e}}(1)}}. \tag{14}$$

1.3 Linearization and Embedding Matrices

The axiom of superlogarithms is a special case of an equation first discovered by Abel [?].

Definition 15 (Abel Linearization Equation)**.** If $A[f](f(z)) = A[f](z) + 1$, then $A[f](z)$ is an **Abel linearization function** for $f(z)$.

Abel found that if a function $A[f](z)$ can be found that satisfies (15), then continuous iteration of a function $f(z)$ could be found by: $f^t(z) = A^{-1}[f](A[f](z) + t)$. (15) is part of a larger family of techniques used for continuous iteration. Abel himself did not find such a function in general, but other methods [?,?] can be used find an Abel linearization function *indirectly*. The method described in this work, however, is an attempt to find an Abel linearization function *directly*.

Consider the exponential of the Abel linearization equation (15) base λ:

$$\lambda^{A[f](f(z))} = \lambda^{A[f](z)+1} = \lambda \cdot \lambda^{A[f](z)}. \tag{16}$$

This has become known as the Schröder linearization equation.

Definition 17 (Schröder Linearization Equation)**.** If $S[f](f(z)) = \lambda S[f](z)$, then $S[f](z)$ is a **Schröder linearization function** for $f(z)$.

Comparing (16) and (17) we find:

$$S[f](z) = \lambda^{A[f](z)}, \tag{18}$$

where λ is the eigenvalue of (17). Eigenvalues only apply to linear equations (which (17) is not), which leads us to embedding matrices. E. T. Bell [?] was first to discover **embedding matrices** which were later generalized by Carleman [?]. Embedding matrices effectively *linearize* a nonlinear equation, so they have also been called **linearization matrices**.

Definition 19 (Bell Embedding Matrix)**.** If $f(0) = 0$, the matrix $\mathbf{B}[f]$ where $j, k \geq 1$ is:

$$B_{jk}[f] = \frac{1}{j!}\left[\frac{\partial^j}{\partial z^j}(f(z))^k\right]_{z=0}.$$

Definition 20 (Carleman Embedding Matrix)**.** The matrix $\mathbf{M}[f]$ where $j, k \geq 0$ is:

$$M_{jk}[f] = \frac{1}{k!}\left[\frac{\partial^k}{\partial z^k}(f(z))^j\right]_{z=0}.$$

From these definitions it should be easy to see that if $f(0) = 0$, then $\mathbf{B}^{\mathrm{T}}[f] = \mathbf{M}[f]$. Since both types of embedding matrices are basically transposes of each other, the more general Carleman embedding matrices will be used.

What might not be so obvious are some properties of Carleman embedding matrices:

$$\mathbf{M}[f \circ g] = \mathbf{M}[f]\mathbf{M}[g] \tag{21}$$
$$\mathbf{M}[f^t] = \mathbf{M}^t[f] \tag{22}$$
$$\mathbf{M}_z[cf(z)] = \mathbf{C}\mathbf{M}[f] \text{ where } C_{jk} = c^k \delta_{jk} \tag{23}$$

where $\mathbf{M}_z[f(z)] = \mathbf{M}[f]$, $(\mathbf{M})_{jk} = M_{jk}$ and δ_{jk} is Kronecker delta.

To overview, the previous techniques can be used for continuous iteration as follows:

$$f^t(z) = A^{-1}[f](A[f](z) + t) \tag{24}$$
$$f^t(z) = S^{-1}[f](\lambda^t S[f](z)) \tag{25}$$
$$f^t(z) = \sum_{k=1}^{\infty} (\mathbf{B}^t[f])_{k1} z^k \text{ where } f(0) = 0 \tag{26}$$
$$f^t(z) = \sum_{k=0}^{\infty} (\mathbf{M}^t[f])_{1k} z^k \tag{27}$$

So Carleman embedding matrices effectively turn the problem of continuous iteration into a problem of evaluating a matrix power. The problem of evaluating a matrix power in turn can be posed as a problem of matrix diagonalizaiton:

$$\mathbf{M}^t[f] = \mathbf{U}^{-1}\mathbf{\Lambda}^t\mathbf{U} \tag{28}$$
$$\mathbf{M}[f] = \mathbf{U}^{-1}\mathbf{\Lambda}\mathbf{U} \tag{29}$$
$$\mathbf{U}\mathbf{M}[f] = \mathbf{\Lambda}\mathbf{U} \tag{30}$$

in which the rows of $\mathbf{U}$ are the left-eigenvectors and $\Lambda_{jk} = \lambda^k \delta_{jk}$ is a diagonal matrix of the eigenvalues. After taking the Carleman embedding matrix of (17):

$$\mathbf{M}_z[S[f](f(z)] = \mathbf{M}_z[\lambda S[f](z)] \tag{31}$$
$$\mathbf{M}[S[f]]\mathbf{M}[f] = \mathbf{\Lambda}\mathbf{M}[S[f]] \tag{32}$$

it is found that $\mathbf{U} = \mathbf{M}[S[f]]$ in (30), and that $\lambda = f'(z_0)$ where $f(z_0) = z_0$ [?].

2 Results

2.1 Solving The Abel Linearization Equation

Following the Abel linearization equation to its ultimate end, and asserting that the Abel linearizing function is infinitely differentiable, an infinite system of linear equations can be constructed. This system is represented by what the author calls a *Robbins matrix.*

To derive the Robbins matrix, we first assume a Taylor series expansion for $A[f](z)$:

$$A[f](z) = \sum_{k=0}^{\infty} c_k (z - z_0)^k. \tag{33}$$

Definition 34 (Robbins Matrix)**.** Let the matrix $\mathbf{R}[f](z_0)$ where:

$$\boxed{R_{jk}[f](z_0) = \left[\frac{\partial^{(j-1)}}{\partial z^{(j-1)}} \left((f(z) - z_0)^k - (z - z_0)^k\right)\right]_{z=z_0}}$$

be the **Robbins matrix** for $f(z)$ where $j, k \in [1, \infty)$. Let the matrix $\mathbf{R}[f](z_0)_{nm}$ where: $R_{jk}[f](z_0)_{nm} = (R_{jk}[f](z_0))(1 - \delta_{mk}) + \delta_{j1}\delta_{mk}$ be an $n \times n$ **truncated Robbins matrix** for $f(z)$ where $j, k \in [1, n]$ and δ_{jk} is Kronecker delta. Note that $\mathbf{R}[f](z_0) = \mathbf{R}[f](z_0)_{\infty 0}$.

Also note: $\mathbf{R}[f](0) = \mathbf{J}(\mathbf{B}[f] - \mathbf{I}) = \mathbf{J}(\mathbf{M}^{\mathrm{T}}[f] - \mathbf{I})$ where $J_{jk} = k!\delta_{(j-1)k}$.

Theorem 35 (Abel Linearizing Function)**.** *Given $f(z)$ and a point $(z_0, A[f](z_0))$:*

$$\boxed{A[f](z) = A[f](z_0) + \sum_{k=1}^{\infty} \frac{\det(\mathbf{R}[f](z_0)_{\infty k})}{\det(\mathbf{R}[f](z_0)_{\infty 0})} (z - z_0)^k}$$

is an Abel linearizing function of $f(z)$ where $\det(\mathbf{R}[f](z_0)) \neq 0$, the limit:

$$c_k = \lim_{n \to \infty} \frac{\det(\mathbf{R}[f](z_0)_{nk})}{\det(\mathbf{R}[f](z_0)_{n0})}$$

exists, and the series converges over some open ball around $z = z_0$.

Proof of (35). Start with the Abel linearization equation (15) rearranged so that the 1 is by itself, and take the mth derivative of both sides:

$$1 = A[f](f(z)) - A[f](z) \tag{36}$$

$$\frac{\partial^m}{\partial z^m}(1) = \frac{\partial^m}{\partial z^m} \left(A[f](f(z)) - A[f](z)\right) \tag{37}$$

$$\delta_{m0} = \frac{\partial^m}{\partial z^m} A[f](f(z)) - \frac{\partial^m}{\partial z^m} A[f](z) \tag{38}$$

replace $A[f](z)$ with the Taylor expansion (33) and simplify:

$$\delta_{m0} = \frac{\partial^m}{\partial z^m} \sum_{k=0}^{\infty} c_k (f(z) - z_0)^k - \frac{\partial^m}{\partial z^m} \sum_{k=0}^{\infty} c_k (z - z_0)^k \tag{39}$$

$$= \sum_{k=0}^{\infty} c_k \frac{\partial^m}{\partial z^m} (f(z) - z_0)^k - \sum_{k=0}^{\infty} c_k \frac{\partial^m}{\partial z^m} (z - z_0)^k \tag{40}$$

$$= \sum_{k=0}^{\infty} c_k \frac{\partial^m}{\partial z^m} \left((f(z) - z_0)^k - (z - z_0)^k\right) \tag{41}$$

at this point it is obvious that for $m \geq 0$, the equations (41) are linear in c_k. Let this system of linear equations be denoted by the matrix equation:

$$\mathbf{b} = \mathbf{R}\mathbf{c}. \tag{42}$$

where $\mathbf{c} = \{c_k\}_{k=0}^{\infty}$ and $\mathbf{b} = \{\delta_{m0}\}_{m=0}^{\infty}$ (the first column of the identity matrix). Now, it should be noted that if we replace k with 0 in (41), that the coefficient of c_0 is 0 for all m. This means that c_0 is not involved in these equations, and thus must be given.

If we are expanding $A[f](z)$ around the known point $(z_0, A[f](z_0))$, then we must evaluate these equations at $z = z_0$ to make $A[f](z)$ differentiable at that point (for proof, see (?)). Removing c_0 from (41), replacing m with $(j-1)$, and evaluating each side at $z = z_0$:

$$\begin{aligned}
\delta_{m0} &= 0c_0 + \sum_{k=1}^{\infty} c_k \frac{\partial^m}{\partial z^m} \left((f(z) - z_0)^k - (z - z_0)^k \right) && (43)\\
\delta_{j1} &= \sum_{k=1}^{\infty} c_k \frac{\partial^{(j-1)}}{\partial z^{(j-1)}} \left((f(z) - z_0)^k - (z - z_0)^k \right) && (44)\\
&= \left[\sum_{k=1}^{\infty} c_k \frac{\partial^{(j-1)}}{\partial z^{(j-1)}} \left((f(z) - z_0)^k - (z - z_0)^k \right) \right]_{z=z0} && (45)\\
&= \sum_{k=1}^{\infty} c_k \left[\frac{\partial^{(j-1)}}{\partial z^{(j-1)}} \left((f(z) - z_0)^k - (z - z_0)^k \right) \right]_{z=z0} && (46)\\
&= \sum_{k=1}^{\infty} c_k R_{jk}[f](z_0) && (47)\\
\mathbf{b} &= \mathbf{R}[f](z_0)\mathbf{c} && (48)
\end{aligned}$$

where $\mathbf{c} = \{c_k\}_{k=1}^{\infty}$ (the coefficients in (33)), $\mathbf{b} = \{\delta_{j1}\}_{j=1}^{\infty}$ and $\mathbf{R}[f](z_0)$ is the Robbins matrix (34). Note the similarities and slight differences between (42) and (48).

Having a well-specified system of linear equations we can now solve for c_k for all $k \geq 1$ with the help of Cramer's rule [?]. The coefficients c_k are the quotient of the determinants of the Cramer's rule matrices and the determinant of the Robbins matrix. Furthermore, the Cramer's rule matrices are embedded in truncated Robbins matrices (34) and are just $\mathbf{R}[f](z_0)_{\infty k}$. This means that the coefficients can be found by:

$$c_k = \lim_{n \to \infty} \frac{\det(\mathbf{R}[f](z_0)_{nk})}{\det(\mathbf{R}[f](z_0)_{n0})}. \tag{49}$$

provided the limit exists. Knowing the c_k in (33), the $A[f](z)$ in (35) is the Abel linearizing function of a function $f(z)$ if and only if the series converges within some open ball. □

Knowing the Abel linearization function, this means the superlogarithm has been found.

2.2 Analytic Continuation of Superlogarithms

Theorem 50. $\mathrm{slog}_x(z)$ *is the Abel linearization function of exponentials:* $A[\exp_x](z)$.

Proof of (50). In (15), replace $f(z)$ with x^z and $A[f](z)$ with $\mathrm{slog}_x(z)$, then subtract 1 to get (9). $\mathrm{slog}_x(z)$ satisfies (15), thus $\mathrm{slog}_x(z) = A[\exp_x](z)$. □

Definition 51 (Robbins Matrix for $f(z) = x^z$)**.** Let the respective matrices:

$$\mathbf{S}(x, z_0) = \mathbf{R}[\exp_x](z_0),$$
$$\mathbf{S}(x, z_0)_{nm} = \mathbf{R}[\exp_x](z_0)_{nm},$$

be the Robbins matrix and truncated Robbins matrix for $f(z) = x^z$ around the point z_0.

The matrix $\mathbf{S}(x, z_0)$ can now be simplified, since the general derivatives are known:

$$\boxed{S_{jk}(x, z_0) = \sum_{i=0}^{k} \binom{k}{i} (-z_0)^{k-i} \left(x^{iz_0} (i \log(x))^{(j-1)} - \frac{i! z_0^{i-(j-1)}}{(i-(j-1))!} \right)} \tag{52}$$

where $0^0 = 1$, and more simplified forms for $z_0 = 0$ and $x = \boldsymbol{e}$:

$$S_{jk}(x, 0) = (k \log(x))^{(j-1)} - k!\delta_{(j-1)k} \tag{53}$$
$$S_{jk}(\boldsymbol{e}, 0) = k^{(j-1)} - k!\delta_{(j-1)k} \tag{54}$$

The last form (54) is almost a Vandermonde matrix [?]. The matrix is of the form $\mathbf{V} - \mathbf{J}$ where $V_{jk} = k^{(j-1)}$ is a Vandermonde matrix and $J_{jk} = k!\delta_{(j-1)k}$ is an off-diagonal matrix. This knowledge will be useful later.

Definition 55 (Coefficient Approximations)**.** Let:

$$\boxed{s_k(x, z_0)_n = \frac{\det(\mathbf{S}(x, z_0)_{nk})}{\det(\mathbf{S}(x, z_0)_{n0})}}$$

be the n-th approximation to the k-th coefficient of the superlogarithm to base-x.

Lemma 56 (Matrix Invertability)**.** $\det(\mathbf{S}(x, z_0)_{n0}) \neq 0$ *for all* n, x *where* $x > 1$.

Proof of (56). The determinants $\det(\mathbf{S}(x, z_0)_{n0})$ are polynomials in $\log(x)$ with positive coefficients. The only way the determinants could be zero, then, is if $\log(x)$ is negative, which only happens if $x < 1$. Thus if $x > 1$, then $\det(\mathbf{S}(x, z_0)_{n0}) \neq 0$. □

Conjecture 57 (Coefficient Signs)**.** *For sufficiently large* n*, the signs of the coefficients* $s_k(\boldsymbol{e}, 0)_n$ *follow a pattern. The signs of any seven coefficients are negative of the next seven coefficients in the series (i.e.:* $s_{k+7}/|s_{k+7}| = -s_k/|s_k|$*). Remembering* $s_0 = -1$*, the signs of the coefficients* $\{s_0, s_1, ..., s_{13}\}$ *for* $x = \boldsymbol{e}$ *are:* $\{-, +, +, -, -, +, +, +, -, -, +, +, -, -\}$.

Lemma 58 (Coefficient Convergence)**.** *The limit* $\lim_{n\to\infty} s_k(x,z_0)_n$ *exists for* $x>1$.

Proof of (58). Using an upper bound for the difference: $|s_k(x,z_0)_{n+1} - s_k(x,z_0)_n| < \boldsymbol{e}^{\left(\frac{k-n}{e}\right)}$ it can be shown that $\lim_{n\to\infty} \boldsymbol{e}^{\left(\frac{k-n}{e}\right)} = 0$ so each coefficient converges to a finite number $\square$

Lemma 59 (Series Convergence)**.** $\sum_{k=1}^{\infty} s_k(x,z_0)_\infty (z-z_0)^k$ *converges if* $|z-z_0| < \boldsymbol{e}^{1/\boldsymbol{e}}$.

Proof of (59). Using an upper bound for coefficients: $|s_k(x,z_0)_\infty| < \boldsymbol{e}^{-\left(\frac{k}{e}+\frac{1}{7}\right)}$ if $k>2$, it can be found that $\lim_{k\to\infty} \boldsymbol{e}^{-\left(\frac{k}{e}+\frac{1}{7}\right)} = 0$. By (57) the series is essentially alternating, so by the *alternating series test* [6], the series converges within some radius.

The radius of convergence can now be found by the limit of the supremum of the roots of the coefficients. An upper bound of the roots is: $\left|s_k(x,z_0)_\infty^{1/k}\right| < \boldsymbol{e}^{-\left(\frac{1}{e}+\frac{1}{7k}\right)}$ if $k>2$, so:

$$c = \lim_{n\to\infty} \sup_{k\geq n} \left|s_k(x,z_0)_\infty^{1/k}\right| < \lim_{n\to\infty} \boldsymbol{e}^{-\left(\frac{1}{e}+\frac{1}{7n}\right)} = \boldsymbol{e}^{-1/\boldsymbol{e}} \tag{60}$$

$$R = \frac{1}{c} > \boldsymbol{e}^{1/\boldsymbol{e}} \tag{61}$$

so the series converges at least where $|z-z_0| < \boldsymbol{e}^{1/\boldsymbol{e}}$. $\square$

Theorem 62 (Superlogarithm)**.** *Given* x *and a point* $(z_0, \text{slog}_x(z_0))$*:*

$$\boxed{\text{slog}_x(z) = \text{slog}_x(z_0) + \sum_{k=1}^{\infty} s_k(x,z_0)_\infty (z-z_0)^k}$$

where $|z-z_0| < \boldsymbol{e}^{1/\boldsymbol{e}}$ *is the superlogarithm to base-*x *of* z *about the point* z_0 *if* $x>1$.

Proof of (62). The matrix is invertible by (56), the coefficients converge by (58), and the series converges within at least a radius of $\boldsymbol{e}^{1/\boldsymbol{e}}$ by (59), thus (62) is the superlogarithm. $\square$

Theorem 63 (Complex Conjugate)**.** $\text{slog}_x(z)$ *satisfies:* $\text{slog}_x(\overline{z}) = \overline{\text{slog}_x(z)}$.

Proof of (63). The complex conjugate commutes over addition and multiplication, so since the series (62) consists of only these, the complex conjugate commutes over it. $\square$

Conjecture 64 (Real Equilibrium Existence)**.** *Functions* $f(x)$ *and* $g(x,u)$ *exist such that:*

$$\text{slog}_x(f(x) + \boldsymbol{i}u) = \text{slog}_x(f(x)) + \boldsymbol{i}g(x,u)$$

where $f(x) \approx \boldsymbol{e}^{2-x}$ *and* $g(x,u) \approx u^3$.

Rough Proof of (64). Take for example $x = \boldsymbol{e}$:

$$\text{slog}_{\boldsymbol{e}}(0.473) \approx -0.527,$$
$$\Re(\text{slog}_{\boldsymbol{e}}(0.473 + \boldsymbol{i}u)) \approx -0.527 \text{ for all } u.$$

where $\Re(z)$ is the real part of the complex number z. $\square$

2.2.1 Piecewise Definition of Superlogarithms

Definition 65 (Superlogarithm Approximations)**.** Let:

$$\boxed{sa(x, z, n, (z_0, y_0)) = y_0 + \sum_{k=1}^{n} s_k(x, z_0)_n (z - z_0)^k}$$

be the n-th approximation of the superlogarithm to base-x of z about the point (z_0, y_0).

Using (65) we can see that (62) could be stated: $\mathrm{slog}_x(z) = sa(x, z, \infty, (z_0, y_0))$.

Theorem 66. $sa(x, z_0, n, (z_0, y_0)) = y_0$ *for all* x, n, z_0, y_0.

Proof of (66). (Trivial.) Replace z with z_0 in (65) and the summation vanishes. □

Theorem 67. $sa(x, x^{z_0}, n, (z_0, y_0)) = y_0 + 1$ *for all* x, n, z_0, y_0.

Rough Proof of (67). $sa(\cdots)$ is a solution to the n-th order system of linear equations that embed the axiom of superlogarithms (9). Since $\mathrm{slog}_x(x^{z_0}) = \mathrm{slog}_x(z_0) + 1$ then any approximation would have to satisfy this as well. Replacing $\mathrm{slog}_x(z_0)$ with y_0 gives (67). □

Having a series for the superlogarithm around any point has its advantages, but is not the most computationally efficient method of calculating superlogarithms. Using (65) and keeping in mind (66-67), we could define superlogarithms over real number z by:

$$\mathrm{slog}_x(z) = \begin{cases} sa(x, x^z, \infty, (0, -1)) - 1 & \text{if } z < 0, \\ sa(x, z, \infty, (0, -1)) & \text{if } 0 \le z < 1, \\ sa(x, z, \infty, (1, 0)) & \text{if } 1 \le z < x, \\ sa(x, z, \infty, ({}^m x, m)) & \text{if } {}^m x \le z < {}^{(m+1)}x, \end{cases} \tag{68}$$

where $m \in \mathbb{Z}^+$. Note that (68) converges over each interval because of (66-67). However, a more computationally efficient method is to define superlogarithms over real number z by:

$$\mathrm{slog}_x(z) = \begin{cases} sa(x, x^z, \infty, (0, -1)) - 1 & \text{if } z < 0, \\ sa(x, z, \infty, (0, -1)) & \text{if } 0 \le z < 1, \\ sa(x, \log_x^m(z), \infty, (0, -1)) + m & \text{if } {}^{(m-1)}x \le z < {}^m x, \end{cases} \tag{69}$$

Equating pieces from each of the piecewise-definitions above gives:

$$sa(x, z, \infty, ({}^{(m-1)}x, m-1)) = sa(x, \log_x^m(z), \infty, (0, -1)) + m, \tag{70}$$

where ${}^{(m-1)}x \le z \le {}^m x$. Since ${}^\infty x$ converges for $\boldsymbol{e}^{-\boldsymbol{e}} \le x \le \boldsymbol{e}^{1/\boldsymbol{e}}$ [?], these definitions only define superlogarithms over all real numbers if $x > \boldsymbol{e}^{1/\boldsymbol{e}}$. If $1 < x \le \boldsymbol{e}^{1/\boldsymbol{e}}$, then not all pieces of the above piecewise-defined functions would converge.

2.2.2 Closed Form Expressions of Superlogarithms

For certain bases, a closed form expression for the superlogarithm can be found, but before we can do this, a closer look at the Robbins matrices for $f(z) = x^z$ are needed.

The expressions that follow require the use of the superfactorial. The **superfactorial** is defined as $n\$ = \prod_{k=1}^{n} k!$ [?], and the Barnes G-function is defined over complex numbers such that $n\$ = G(n+2)$ [?]. In the interest of generality, the following equations will use the Barnes G-function instead of the superfactorial. The determinants of the Robbins matrices for $f(z) = x^z$ are polynomials in $\log(x)$, where the coefficients grow like the superfactorial.

Definition 71 (Determinant Coefficient Matrix)**.** Let $\mathbf{C}(n)$ be the matrix that satisfies:

$$\det(\mathbf{S}(x,0)_{nk}) = G(n+1) \sum_{j=0}^{\binom{n}{2}} C_{jk}(n) \log(x)^j$$

for all $0 \le k \le n$, where the dimensions of $\mathbf{C}(n)$ are $(\binom{n}{2}+1) \times (n+1)$ and $\binom{n}{2} = n(n-1)/2$ is the highest power of $\log(x)$ (the sum of $1 + \cdots + (n-1)$, the powers of $\log(x)$).

Examples of the matrix $\mathbf{C}(n)$ are given for $n = 1$ and $n = 2$, see appendix (?) for more:

$$\mathbf{C}(1) = \begin{pmatrix} 1 & 1 \end{pmatrix} \tag{72}$$

$$\mathbf{C}(2) = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 2 & -1 \end{pmatrix} \tag{73}$$

Defining $\mathbf{C}(n)$ where $z_0 = 0$ means we can use $\mathbf{C}(n)$ to expand the superlogarithm about the point $(0, -1)$, thus (72-73) correspond to the superlogarithm approximations:

$$sa(x, z, 1, (0, -1)) = -1 + \frac{C_{01}(1)}{C_{00}(1)} z \tag{74}$$

$$= -1 + z \text{ where } 0 \le z \le 1 \tag{75}$$

$$sa(x, z, 2, (0, -1)) = -1 + \frac{C_{01}(2) + C_{11}(2)\log(x)}{C_{00}(2) + C_{10}(2)\log(x)} z + \frac{C_{02}(2) + C_{12}(2)\log(x)}{C_{00}(2) + C_{10}(2)\log(x)} z^2 \tag{76}$$

$$= -1 + \frac{2\log(x)}{1 + \log(x)} z + \frac{1 - \log(x)}{1 + \log(x)} z^2 \text{ where } 0 \le z \le 1 \tag{77}$$

and in general:

$$sa(x, z, n, (0, -1)) = -1 + \sum_{k=1}^{n} \frac{\sum_{j=0}^{\binom{n}{2}} C_{jk}(n) \log(x)^j}{\sum_{j=0}^{\binom{n}{2}} C_{j0}(n) \log(x)^j} z^k \tag{78}$$

where $0 \le z \le 1$.

Theorem 79 (Determinant Summability)**.** $\det(\mathbf{R}[f](z_0)_{n0}) = \sum_{k=1}^{n} \det(\mathbf{R}[f](z_0)_{nk})$.

Proof of (79). Since the vector $\mathbf{b}$ being replaced in the Cramer's rule matrices $\mathbf{R}[f](z_0)_{nk}$ is the first column of the identity matrix, by the evaluation of a determinant by minors, the Cramer's rule matrices are simply the minors of the determinant of $\mathbf{R}[f](z_0)_{n0}$, and since the sign is embeded in $\det(\mathbf{R}[f](z_0)_{nm})$, their sum is $\det(\mathbf{R}[f](z_0)_{n0})$. □

Applying (79) to $\mathbf{C}(n)$ we get: $C_{j0}(n) = \sum_{k=1}^{n} C_{jk}(n)$ for all $0 \le j \le N$. With (79), further proof of (67) can now be provided rearranging (78):

More Proof of (67). By (79), replace $C_{j0}(n)$ with $\sum_{k=1}^{n} C_{jk}(n)$ in (78):

$$sa(x, z, n, (0, -1)) = -1 + \frac{\sum_{k=1}^{n} \sum_{j=0}^{\binom{n}{2}} C_{jk}(n) \log(x)^j z^k}{\sum_{j=0}^{\binom{n}{2}} \sum_{k=1}^{n} C_{jk}(n) \log(x)^j} \tag{80}$$

$$= -1 + \frac{F(x, z, n)}{F(x, 1, n)} \tag{81}$$

where $0 \le z \le 1$. Now it is easy to see that (67) is true if $z = 1$ and $z_0 = 0$ for all x, n. Proof would be similar for other z_0 (and also for other $\mathbf{R}[f](z_0)$). □

Before continuing, recall that the coefficients are not dependant on a specific multiple of the determinants, but could be found from any definite multiple ($X \ne 0$):

$$s_k(x, z_0)_n = \frac{\det(\mathbf{S}(x, z_0)_{nk})}{\det(\mathbf{S}(x, z_0)_{n0})} = \frac{\det(\mathbf{S}(x, z_0)_{nk})X}{\det(\mathbf{S}(x, z_0)_{n0})X} \tag{82}$$

Lemma 83. $\det(\mathbf{S}(x, 0)_{nk}) \log(x)^{-\binom{n}{2}}/G(n+1) = \sum_{j=0}^{\binom{n}{2}} C_{(\binom{n}{2}-j)k}(n) \log(x)^{-k}$.

Proof of (83). Dividing (71) by $\log(x)^{\binom{n}{2}}$ reverses the order of the determinant coefficients becuase $\binom{n}{2}$ is the highest power of $\log(x)$. □

Lemma 84. $\lim_{x\to\infty} \det(\mathbf{S}(x, 0)_{nm})/\log(x)^N = C_{Nm}(n)$ *where* $N = n(n-1)/2$.

Proof of (84). Replacing x with ∞ in (83) gives the only constant coefficient. □

Lemma 85. $\det(\mathbf{L}(x)\mathbf{S}(x, 0)_{nm}) = \det(\mathbf{S}(x, 0)_{nm})/\log(x)^N$ *where* $L_{jk}(x) = \delta_{jk}/\log(x)^{(j-1)}$.

Proof of (85). $\det(\mathbf{L}(x))$ is the product of its diagonal entries which would be $1/\log(x)^N$ where $N = n(n-1)/2$. Multiplying this by $\det(\mathbf{S}(x, 0)_{nm})$ gives (85) □

Lemma 86. $\mathbf{L}(x)\mathbf{S}(x, 0) = \mathbf{V}' - \mathbf{J}'(x)$ *where* $V'_{jk} = k^{(j-1)}$, $J'_{jk} = k!\delta_{(j-1)k}/\log(x)^{(j-1)}$, *and* $L_{jk}(x) = \delta_{jk}/\log(x)^{(j-1)}$.

Proof of (86). (Trivial.) Subtract $\mathbf{J}'(x)$ from $\mathbf{V}'$ to test. □

Lemma 87. $\lim_{x\to\infty} \mathbf{L}(x)\mathbf{S}(x,0) = \mathbf{V}'$ *where* $V'_{jk} = k^{(j-1)}$, *and* $L_{jk}(x) = \delta_{jk}/\log(x)^{(j-1)}$.

Proof of (87). As x tends to ∞, then $1/\log(\infty) = 0$ so $\mathbf{J}'(\infty)$ vanishes. □

Theorem 88 (Top Row of $\mathbf{C}$)**.** $C_{0k}(n) = \delta_{nk}$ *for all* $1 \le k \le n$.

Proof of (88). Start by noticing that if $x = 1$ in (71) then $\det(\mathbf{S}(1,0)_{nk}) = C_{0k}(n)$, which is the only constant coefficient. Recall from (**??**) that:

$$\mathbf{S}(x,0) = \mathbf{V}(x) - \mathbf{J}, \tag{89}$$

where $V_{jk}(x) = (k\log(x))^{(j-1)}$, $J_{jk} = k!\delta_{(j-1)k}$ and $j,k \ge 1$. If $x = 1$ in (89), then $\mathbf{V}(1)$ vanishes because $\log(1) = 0$. So:

$$\mathbf{S}(1,0) = \mathbf{J}, \tag{90}$$

where $J_{jk} = k!\delta_{(j-1)k}$ and $j,k \ge 1$.

Case 1, where $(1 \le k \le (n-1))$:

$$\det(\mathbf{S}(1,0)_{nm}) = \det(\mathbf{J}) = 0. \tag{91}$$

The Cramer's rule matrices $\mathbf{S}(1,0)_{nk}$ where $1 \le k \le (n-1)$ effectively replace a column of $\mathbf{J}$ with the first column of the identity matrix and since there was one row and one column with all zeros in $\mathbf{J}$ to begin with (implying a zero determinant), the determinant will be zero after replacement as well.

Case 2, where $k = n$:

$$\det(\mathbf{S}(1,0)_{nn}) = \det(\mathbf{J}'') = (n-1)\$ = G(n+1), \tag{92}$$

where $J''_{jk} = k!\delta_{(j-1)k} + \delta_{j1}\delta_{nk}$. The Cramer's rule matrices $\mathbf{S}(1,0)_{nn}$ replace the last column (the column with all zeros) with the first column of the identity matrix, which makes $\mathbf{J}''$ what could be called a rotated-diagonal matrix, which means its determinant is just the product of its entries. Since its entries are factorials, its determinant is a superfactorial, or $(n-1)\$ = G(n+1)$ since $\mathbf{J}''$ contains the factorials $0!$ upto $(n-1)!$. Remember that these superfactorials are already embeded in the definition of $\mathbf{C}(n)$, so $C_{0n}(n) = 1$.

Since both cases have been proven the combined form has been proven as well. □

Lemma 93 (Bottom Row of $\mathbf{C}$)**.** $C_{Nm}(n) = G(n+1)(-1)^{m-1}\binom{n}{m}$ *for all* $1 \le m \le n$.

Rough Proof of (93). . □

Theorem 94 (Corners of $\mathbf{C}$)**.** *The corners of* $\mathbf{C}(n)$ *are superfactorials. In other words:* $C_{00}(n) = C_{0n}(n) = C_{N0}(n) = |C_{Nn}(n)| = (n-1)\$ = G(n+1)$ *where* $N = n(n-1)/2$, $n\$$ *is the superfactorial [?] and* $G(n)$ *is the Barnes G-function [?].*

Proof of (94). It is well-known that the determinants of Vandermonde matrices (54) are superfactorials [?]. Recall that:

Since $\mathbf{V}(1) = \mathbf{0}$

and $\mathbf{J}'(\infty) = \mathbf{0}$,

If $x = \boldsymbol{e}$, then $\log(x) = 1$ If $x = 1$, then $\log(x)$ vanishes, and (71) becomes: $\det(\mathbf{S}(1,0)_{nm}) = C_{0m}(n)$. Since $\mathbf{S}(1,0)_{nm}$ is a Vandermonde matrix, then

and FIXME ☐

Conjecture 95 (Diagonals of $\mathbf{C}$). $C_{kk}(n) = G(n+1)\frac{n^k}{k!}$ *for all* $0 \le k \le (n-1)$.

Conjecture 96 (Last Column of $\mathbf{C}$). $C_{kn}(n) = G(n+1)\frac{a(k)}{k!}$ *for all* $0 \le k \le (n-1)$ *where* $\{a(k)\}_{k=0}^{n-1}$ *is the sequence:* $\{1, -1, -4, -21, -160, -1505, -17136, ...\}$.

Conjecture 97 (Zeros of $\mathbf{C}$). $C_{km}(n) = 0$ *for all* $0 \le k \le (m-1)$ *and* $(k+1) \le m \le (n-1)$.

Rough Proof of (97). This is true for the top row by (**??**,88), for examples, see (73) or (129-131) in appendix (?). ☐

Conjecture 98. *Using the Barnes G-function* $(G(n))$*, Lagrange interpolating polynomials, and finite differences, the off-diagonal elements of* $\mathbf{C}(n)$ *can be parameterized as:*

$$C_{mm}(n) = G(n+1)\frac{n^m}{m!} \tag{99}$$

$$C_{(m+1)m}(n) = G(n+1)\frac{n - n^m}{m!} \tag{100}$$

$$C_{(m+2)m}(n) = G(n+1)\frac{2^{m-1}n^2 - 2n^m}{m!} \tag{101}$$

$$C_{(m+3)m}(n) = G(n+1)\frac{(2^m - 2^2)n - (2^m - 2)n^2 + 3^{m-1}n^3 - 7n^m}{2!m!} \tag{102}$$

$$C_{(m+4)m}(n) = G(n+1)\frac{(3^m - 3^3)n - (3^m - 3)n^3 + 4^{m-1}n^4 - 40n^m}{3!m!} \tag{103}$$

$$C_{(m+5)m}(n) = G(n+1)\frac{(4^m - 4^4)n + \cdots - (4^m - 4)n^4 + 5^{m-1}n^5 - 301n^m}{4!m!} \tag{104}$$

$$\vdots \tag{105}$$

$$C_{(m+p)m}(n) = G(n+1)\left(\frac{A_p}{m!}n^m + \sum_{q=1}^{p}\frac{B_q(m+p,m)}{p!m!}n^q\right) \tag{106}$$

where:

$$\{A_k\}_{k=0}^{6} = \{1, -1, -2, -\frac{7}{2}, -\frac{40}{3!}, -\frac{301}{4!}, -\frac{2856}{5!}\} \tag{107}$$

$$A_k = C_{kn}(n) \tag{108}$$

$$B_1(p,m) = p((p-1)^m - {}^2(p-1)) \tag{109}$$

$$B_{p-1}(p,m) = -p((p-1)^m - (p-1)) \tag{110}$$

$$B_p(p,m) = p^m \tag{111}$$

$$\tag{112}$$

Theorem 113 (Superlogarithms to Base-1)**.** *Where* $0 \leq z \leq 1$*:*

$$\boxed{\mathrm{slog}_1(z) = \lim_{n\to\infty}(z^n - 1) = \begin{cases} -1 & \text{if } 0 \leq z < 1, \\ 0 & \text{if } z = 1. \end{cases}}$$

Proof of (113). . □

Theorem 114 (Superlogarithms to Base-∞))**.** *Where* $0 \leq z \leq 1$*:*

$$\boxed{\mathrm{slog}_\infty(z) = \lim_{n\to\infty} -(1-z)^n = \begin{cases} -1 & \text{if } z = 0, \\ 0 & \text{if } 0 < z \leq 1. \end{cases}}$$

Proof of (114). . □

Definition 115 (Sequence)**.** Let $\mathbf{T}(x)_n$ be:

$$T_{jk}(x)_n = \delta_{jk} - \frac{(kx)^j}{j!} \tag{116}$$

then:

$$\det(\mathbf{T}(x)_n) = \sum_{k=0}^{n} a_k x^k + \cdots \tag{117}$$

$$\det(\mathbf{T}(x)) = 1 - x - 2x^2 - \frac{7}{2}x^3 - \frac{20}{3}x^4 - \frac{301}{24}x^5 + \cdots \tag{118}$$

2.3 Analytic Continuation of Tetration

Using the previous definitions of the superlogarithm we can now extend general iterated exponentials to complex number iterations by (15,24,35,62):

$$\boxed{\exp_x^y(a) = \mathrm{slog}_x^{-1}(\mathrm{slog}_x(a) + y)} \tag{119}$$

for all $\boldsymbol{e}^{1/\boldsymbol{e}} < x \in \mathbb{R}$ and $y, a \in \mathbb{C}$, which naturally extends tetration as:

$$
\begin{aligned}
{}^{y}x &= \exp_x^y(1) && (120)\\
&= \operatorname{slog}_x^{-1}(\operatorname{slog}_x(1) + y) && (121)\\
&= \operatorname{slog}_x^{-1}(0 + y) && (122)\\
&= \operatorname{slog}_x^{-1}(y) && (123)
\end{aligned}
$$

2.4 Specific Values of ${}^{y}\boldsymbol{e}$

2.4.1 On ${}^{\pi}\boldsymbol{e}$

$$
{}^{\pi}\boldsymbol{e} = \operatorname{slog}_{\boldsymbol{e}}^{-1}(\pi) \approx 3.17 \times 10^{10} \tag{124}
$$

2.4.2 On ${}^{\boldsymbol{i}}\boldsymbol{e}$

$$
{}^{\boldsymbol{i}}\boldsymbol{e} = \operatorname{slog}_{\boldsymbol{e}}^{-1}(\boldsymbol{i}) \approx 0.786 + \boldsymbol{i}0.916 \tag{125}
$$

2.4.3 On $\operatorname{slog}_{\boldsymbol{e}}(\boldsymbol{i})$

$$
{}^{\boldsymbol{i}}\boldsymbol{e} = \operatorname{slog}_{\boldsymbol{e}}^{-1}(\boldsymbol{i}) \approx 0.786 + \boldsymbol{i}0.916 \tag{126}
$$

$$
\begin{aligned}
\operatorname{slog}_{\boldsymbol{e}}(\boldsymbol{i}) &\approx -1.394 + \boldsymbol{i}1.017 && (127)\\
\operatorname{slog}_{\boldsymbol{e}}(\boldsymbol{i}) &\approx -\boldsymbol{e}^{1/3} + \boldsymbol{i}\frac{\pi}{3} && (128)
\end{aligned}
$$

3 Determinant Coefficient Matrix Appendix

These are a few more examples of the determinant coefficient matrix $\mathbf{C}(n)$:

$$\mathbf{C}(3) = 2\begin{pmatrix} 1 & 0 & 0 & 1 \\ 2 & 3 & 0 & -1 \\ 5/2 & 0 & 9/2 & -2 \\ 1 & 3 & -3 & 1 \end{pmatrix} \tag{129}$$

$$\mathbf{C}(4) = 12\begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 3 & 4 & 0 & 0 & -1 \\ 6 & 0 & 8 & 0 & -2 \\ 55/6 & 8 & -6 & 32/3 & -7/2 \\ 8 & 14 & 0 & -10 & 4 \\ 13/3 & 0 & 12 & -32/3 & 3 \\ 1 & 4 & -6 & 4 & -1 \end{pmatrix} \tag{130}$$

$$\mathbf{C}(5) = 288\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 4 & 5 & 0 & 0 & 0 & -1 \\ 21/2 & 0 & 25/2 & 0 & 0 & -2 \\ 67/3 & 15 & -10 & 125/6 & 0 & -7/2 \\ 315/8 & 40 & 0 & -20 & 625/24 & -20/3 \\ 51 & 305/6 & 75/2 & -25 & -155/6 & 27/2 \\ 205/4 & 15 & 55 & 10 & -175/4 & 15 \\ 2785/72 & 55 & -50 & 500/9 & -215/8 & 5 \\ 59/3 & 235/6 & 0 & -145/3 & 115/3 & -19/2 \\ 77/12 & 0 & 25 & -100/3 & 75/4 & -4 \\ 1 & 5 & -10 & 10 & -5 & 1 \end{pmatrix} \tag{131}$$

and in general:

$$\mathbf{C}(n) = G(n+1)\begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 1 \\ n-1 & n & 0 & 0 & \cdots & -1 \\ \sum_m C_{2m}(n) & 0 & n^2/2 & 0 & \cdots & -2 \\ \sum_m C_{3m}(n) & n(n-2) & -n(n-1)/2 & n^3/6 & \cdots & -7/2 \\ \sum_m C_{4m}(n) & (n+3)n(n-3)/2 & 0 & -(n+1)n(n-1)/6 & \cdots & -20/3 \\ \sum_m C_{5m}(n) & ? & 0 & ? & \cdots & -301/24 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \sum_m C_{km}(n) & \frac{n^{k-1}-n(k-1)^{k-2}}{(k-2)!} & ? & ? & \cdots & ? \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \sum_m C_{(N-2)m}(n) & ? & 0 & ? & \cdots & ? \\ \sum_m C_{(N-1)m}(n) & 0 & n^2(n-1)/4 & -n^2(n-1)(n-2)/9 & \cdots & (-1)^n(n-1) \\ 1 & \binom{n}{1} & -\binom{n}{2} & \binom{n}{3} & \cdots & (-1)^{n-1} \end{pmatrix} \tag{132}$$

3.1 Something Else

$${}^{\boldsymbol{i}}\boldsymbol{e} = \mathrm{slog}_{\boldsymbol{e}}^{-1}(\boldsymbol{i}) \approx \sqrt{\phi - 1} + \boldsymbol{i}\,\mathrm{slog}'_{\boldsymbol{e}}(0) \tag{133}$$

Taking the derivatives even further:

$$\mathrm{slog}_x^{(n)}(0) = \sum_{k=1}^{n} \binom{n-1}{k-1} \log(x)^{n-k+1}\,\mathrm{slog}_x^{(k)}(1) \tag{134}$$

$$\mathrm{slog}_x^{(n)}(1) = \sum_{k=1}^{n} (-1)^{n-k} \binom{n-1}{k-1} \log(x)^{n-k-1}\,\mathrm{slog}_x^{(k)}(0) \tag{135}$$

and

$$\mathrm{slog}_x(z) = \begin{cases} s(x,-1,0,x^z)-1 & \text{if } -\infty < z \le 0 \\ s(x,-1,0,z) & \text{if } 0 < z \le 1 \\ s(x,0,1,z) & \text{if } 1 < z \le x \\ s(x,1,x,z) & \text{if } x < z \le {}^2x \\ s(x,2,x^x,z) & \text{if } {}^2x < z \le {}^3x \\ \vdots & \end{cases} \tag{136}$$

$$\operatorname{slog}_x(z) = {}_{(0,-1)}\operatorname{slog}_x(z) \text{ where } 0 < z \le 1 \tag{137}$$
$$\operatorname{slog}_{(x,0,1)}(z) = \operatorname{slog}_{(x,-1,0)}(\log_x(z)) + 1 \text{ where } 1 < z \le x \tag{138}$$
$$\operatorname{slog}_{(x,1,x)}(z) = \operatorname{slog}_{(x,-1,0)}(\log_x^2(z)) + 2 \text{ where } x < z \le {}^2x \tag{139}$$
$$\operatorname{slog}_{(x,2,x^x)}(z) = \operatorname{slog}_{(x,-1,0)}(\log_x^3(z)) + 3 \text{ where } {}^2x < z \le {}^3x \tag{140}$$

A simpler matrix is $\mathbf{A}(w)_n$ where $w = \frac{1}{\log(x)}$ and $A_{jk}(\frac{1}{\log(x)})_n = S_{jk}(x,0)_{n0}/\log(x)^{(j-1)}$. Two examples of this matrix that illustrates these matrices in general are:

$$\mathbf{A}(w)_3 = \begin{pmatrix} 1 & 1 & 1 \\ 1-w & 2 & 3 \\ 1 & 4-2w^2 & 9 \end{pmatrix}$$
$$\mathbf{A}(w)_4 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1-w & 2 & 3 & 4 \\ 1 & 4-2w^2 & 9 & 16 \\ 1 & 8 & 27-6w^3 & 64 \end{pmatrix}$$

To reduce the amount of differentiation involved in finding $\mathbf{R}$, we can expand (?) using the binomial theorem [?] and simplify as needed:

$$\delta_{m0} = \sum_{k=0}^{\infty} c_k \frac{\partial^m}{\partial z^m} \left(\sum_{i=0}^{k} \binom{k}{i} (-z_0)^{k-i} (f(z))^i - \sum_{i=0}^{k} \binom{k}{i} (-z_0)^{k-i} (z)^i \right) \tag{141}$$
$$= \sum_{k=0}^{\infty} c_k \frac{\partial^m}{\partial z^m} \sum_{i=0}^{k} \binom{k}{i} (-z_0)^{k-i} \left(f(z)^i - z^i \right) \tag{142}$$
$$= \sum_{k=0}^{\infty} c_k \sum_{i=0}^{k} \binom{k}{i} (-z_0)^{k-i} \frac{\partial^m}{\partial z^m} \left(f(z)^i - z^i \right) \tag{143}$$

where δ_{jk} is Kronecker delta.

3.2 Piecewise-Definition of Super-logarithms

Using (8) it is possible to piecewise-define superlogarithms in terms of $s(x, z)$:

$$\mathrm{slog}_x(z)_n = \begin{cases} s(x, x^z)_n - 1 & \text{if } z \leq 0 \\ s(x, z)_n & \text{if } 0 < z \leq 1 \\ s(x, \log_x^m(z))_n + m & \text{if } ({}^{(m-1)}x) < z \leq ({}^m x), m \in \mathbb{Z}^+ \end{cases} \tag{144}$$

$$\mathrm{slog}_x(z) = \mathrm{slog}_x(z)_\infty \tag{145}$$

$$s(x, z)_n = \sum_{k=0}^{n} z^k u_k(x), \tag{146}$$

$$s(x, z) = s(x, z)_\infty. \tag{147}$$

The function $s(x, z)$ is still unknown, though, because the coefficients $u_k(x)$ are unknown. However, by requiring that the superlogarithm is finitely differentiable ($\boldsymbol{C}^n$) with respect to z, a set of equations can be constructed.

3.3 The Critical Function of the Super-logarithm

The function $s(x, z)$ can be called the *critical function* of the superlogarithm, because the piecewise-definition of $\mathrm{slog}_x(z)$ forces all of the z values that get passed to $s(x, z)$ to be within the set (0, 1]. Since ${}^{-1}x = 0$ and ${}^0x = 1$, that means that $\mathrm{slog}_x(0) = -1$ and $\mathrm{slog}_x(1) = 0$, so the superlogarithm maps [0, 1] onto [-1, 0]. To produce proper values for the superlogarithm, its critical function must also obey $s(x, 0) = -1$ and $s(x, 1) = 0$. For a smooth and differentiable superlogarithm, the critical function should be monotonic in its mapping between [0, 1] and [-1, 0]. It is for this reason that the domain and range of $s(x, z)$ are restricted as such:

$$s : \mathbb{R}^+ \times [0, 1] \mapsto [-1, 0]. \tag{148}$$

4 Results

By requiring that $\mathrm{slog}_x(z)$ is $\boldsymbol{C}^n$ with respect to z, a set of $n+1$ equations can be formed by analyzing the possible discontinuity at $z = 0$. In order to do this, a formal representation of the discontinuity between pieces is needed for each derivative.

4.1 The Piecewise Error Transform

The piecewise error transform is presented here to help construct the equations:

$$\boxed{\underset{z\to c}{\Delta^i}\{f(z)\} = \lim_{z\to c^-} \boldsymbol{D}^i f(z) - \lim_{z\to c^+} \boldsymbol{D}^i f(z)}, \tag{149}$$

which represents the magnitude of discontinuity between pieces of a piecewise-defined function, or its derivatives. To ensure that $\text{slog}_x(z)_n$ is $\boldsymbol{C}^{(n-1)}$ this must be zero:

$$\boxed{\underset{z\to 0}{\Delta^i}\{\text{slog}_x(z)_n\} = 0 \text{ for all } 0 \le i < n}. \tag{150}$$

Note that this represents n equations, and that $\text{slog}_x(z)_n$ is defined in terms of $s(x,z)_n$ which contains n unknowns, so its possible that this system of n equations in n unknowns contains at least one, possibly unique, solution.

Remember that ${}^{-1}x = 0$, which implies $\text{slog}_x(0) = -1$ for all x, which implies $s(x,0) = -1$, so it follows that $u_0 = -1$. Also, for the sake of simplicity, unknowns will not be written as functions (i.e. $u_k = u_k(x)$).

4.2 Piecewise Error Transform of the Super-logarithm

From the definition of the piecewise error transform and the definition of the superlogarithm presented here, a general form of the previous equations can be shown:

$$\begin{aligned}
\underset{z\to 0}{\Delta^i}\{\text{slog}_x(z)_n\} &= \boldsymbol{D}_z^i[s(x,x^z)_n - 1]_{z=0} && - \boldsymbol{D}_z^i[s(x,z)_n]_{z=0}, && (151)\\
&= \boldsymbol{D}_z^i\left[\sum_{k=0}^{n} x^{zk}u_k(x) - 1\right]_{z=0} && - \boldsymbol{D}_z^i\left[\sum_{k=0}^{n} z^k u_k(x)\right]_{z=0} && (152)\\
&= \boldsymbol{D}_z^i\left[\sum_{k=1}^{n} x^{zk}u_k(x) - 1\right]_{z=0} && - \boldsymbol{D}_z^i\left[\sum_{k=1}^{n} z^k u_k(x)\right]_{z=0} && (153)\\
&= \left[\sum_{k=1}^{n} (k\log(x))^i x^{zk}u_k(x) - \delta_{i0}\right]_{z=0} && - \left[\sum_{k=i}^{n} i!z^{k-i}u_k(x)\right]_{z=0} && (154)\\
&= \sum_{k=1}^{n} k^i\log(x)^i u_k(x) - \delta_{i0} && - i!u_i(x), && (155)\\
&= \sum_{k=1}^{n}(k^i\log(x)^i - k!\delta_{ik})u_k(x) - \delta_{i0}, && && (156)
\end{aligned}$$

where $0 \le i < n$, δ_{jk} is the Kronecker delta, and in equation (41) only: $0! = 0$.

The equations in (15) can now be rewritten as:

$$\sum_{k=1}^{n}(k^i\log(x)^i - k!\delta_{ik})u_k(x) = \delta_{i0} \text{ for all } 0 \le i < n. \tag{157}$$

Notice that all the equations are linear, and are thus expressible in matrix form. Replace i with $j-1$ in (41) where $1 \le j \le n$, and a general expression for the matrix form of the equations is $\mathbf{Au} = \mathbf{b}$, where:

$$\boxed{A_{jk} = k^{(j-1)} \log(x)^{(j-1)} - k!\delta_{(j-1)k}}, \tag{158}$$

$$u_k = u_k(x), \tag{159}$$

$$\boxed{b_j = \delta_{(j-1)0}}. \tag{160}$$

In order to make the matrices more clear, a variant of the above matrix is shown, $\mathbf{B}$ where $B_{jk} = A_{jk}/\log(x)^{(j-1)}$, so that the equations still hold, and the majority of components are integers. This illustrates the set of equations for $\operatorname{slog}_x(z)_3$:

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 - \frac{1}{\log(x)} & 2 & 3 \\ 1 & 4 - \frac{2}{\log(x)^2} & 9 \end{pmatrix} \mathbf{u} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \tag{161}$$

4.3 Existence, Uniqueness and Convergence

To determine the existence and uniqueness of solutions to the above equations, their determinants will be found. Determinants will be evaluated by enumerating all of the index permutations. As a reminder, the permutation enumeration method for evaluating the determinant of a square $n \times n$ matrix is defined as:

$$\det(\mathbf{A}) = \sum_{p \in \text{perm}_n} \epsilon_p \prod_{j=1}^{n} A_{jp_j} \tag{162}$$

where ϵ_p is the permutation symbol, $\text{perm}_n = \text{perm}(N_1^n)$, $\text{perm}(S)$ is the set of all permutations of the elements of S, and $N_m^n = \{m, (m+1), ..., (n-1), n\}$.

Using the expressions for $\mathbf{A}$ and $\mathbf{B}$, their determinants in general are:

$$\det(\mathbf{A}) = \sum_{p \in \text{perm}_n} \epsilon_p \prod_{j=1}^{n} ((p_j)^{(j-1)} \log(x)^{(j-1)} - (p_j)!\delta_{(j-1)p_j}) \tag{163}$$

$$\det(\mathbf{B}) = \sum_{p \in \text{perm}_n} \epsilon_p \prod_{j=1}^{n} \left((p_j)^{(j-1)} - \frac{(p_j)!\delta_{(j-1)p_j}}{\log(x)^{(j-1)}} \right) \tag{164}$$

$$= \sum_{a \in L} \left(\prod_{k \in a} \frac{k!}{\log(x)^k} \right) \sum_{b \in Q} \epsilon_b \prod_{\substack{c \in R \\ d \in b}} c^d \tag{165}$$

$$= \sum_{k=0}^{n(n-1)/2} \frac{g_k}{\log(x)^k} \tag{166}$$

where L the set of all subsets of N_1^n, $Q = \text{perm}(N_0^{n-1} - \{a\})$, and $R = N_1^n - \{a\}$.

All g_k are positive (see appendix A), which implies: $\det(\mathbf{B}) = 0$ which implies $\log(x) < 0$ which implies $x < 1$, the negation of which would be:

$$x > 1 \implies \det(\mathbf{B}) \neq 0. \tag{167}$$

Since a unique solution exists for $\mathbf{Bu} = \mathbf{b}$ only when $\det(\mathbf{B})$ is nonzero, one exists when $x > 1$. By restricting x to be greater than one, there exists exactly one solution to the set of equations $\mathbf{B}$ represents. Since $\mathbf{B}$ is row-equivalent to $\mathbf{A}$, it too is invertible, so there is one and only one solution to the equations $\mathbf{A}$ represents. $\square$

4.3.1 Convergence of Approximants to the Super-logarithm Series

To show convergence of the approximants $s(x, z)_n$ to the infinite series $s(x, z)_\infty$, an example proof for $x = \boldsymbol{e}$ will be shown. The approximants will converge if the absolute error approaches zero as $n \to \infty$. Since absolute error requires an accurate value to compare to, we will use the $(2n)$th approximant in place of a perfect value. This leads to the definition of absolute error between approximants as such:

$$E_A = |s(x, z)_n - s(x, z)_{2n}| \tag{168}$$

An upper bound for the absolute error when $x = \boldsymbol{e}$ is:

$$\boxed{|s(\boldsymbol{e}, z)_n - s(\boldsymbol{e}, z)_{2n}| < (n+2)^{-\frac{7}{2}}} \quad \text{if } z \in [0, 1], \tag{169}$$

$$\lim_{n\to\infty} (n+2)^{-\frac{7}{2}} = 0, \tag{170}$$

so the approximants $s(\boldsymbol{e}, z)_n$ converge to the series $s(\boldsymbol{e}, z)_\infty$ as $n \to \infty$. $\square$

4.3.2 Convergence of Coefficients of the Super-logarithm Series

To show that the superlogarithm itself converges (or $s(x, z)_\infty$), a better look at the coefficients is needed. For $n > 7$ a pattern emerges in the coefficients: the signs of any seven coefficients are opposite in the next seven coefficients. Remembering $u_0 = -1$, the signs of the coefficients $\{u_0, u_1, ..., u_{13}\}$ for $x = \boldsymbol{e}$ are:

$$\begin{aligned} \{&-, +, +, -, -, +, +, \\ &+, -, -, +, +, -, -\} \end{aligned} \tag{171}$$

which allows the ratio between every seven coefficients to be approximated as:

$$\frac{u_{k+7}(\boldsymbol{e})}{u_k(\boldsymbol{e})} \approx -k^{7(\frac{7}{k})^{\frac{1}{k}}} \tag{172}$$

An upper bound for the coefficients $u_k(\boldsymbol{e})$ is:

$$\boxed{|u_k(\boldsymbol{e})| < \boldsymbol{e}^{-k\left(\frac{1}{e}+\frac{1}{7k}\right)}} \text{ if } k > 2, \tag{173}$$

$$\lim_{k\to\infty} \boldsymbol{e}^{-k\left(\frac{1}{e}+\frac{1}{7k}\right)} = 0, \tag{174}$$

and since the series $s(\boldsymbol{e}, z)_\infty$ has alternating signs every seven coefficients, by the *alternating series test* [6], the series $s(\boldsymbol{e}, z)_\infty$ converges over some interval. □

4.3.3 Radius of Convergence of the Super-logarithm Series

To find the radius of convergence of the series $s(\boldsymbol{e}, z)_\infty$, the limit of the supremum of the roots of the coefficients is needed. An upper bound of the roots is:

$$\boxed{\left|u_k(\boldsymbol{e})^{\frac{1}{k}}\right| < \boldsymbol{e}^{-\left(\frac{1}{e}+\frac{1}{7k}\right)}} \text{ if } k > 2, \tag{175}$$

which implies:

$$c = \lim_{n\to\infty} \sup_{k\geq n} \left|u_k(\boldsymbol{e})^{\frac{1}{k}}\right| < \lim_{n\to\infty} \boldsymbol{e}^{-\left(\frac{1}{e}+\frac{1}{7n}\right)} = \boldsymbol{e}^{-\frac{1}{e}} \tag{176}$$

$$R = \frac{1}{c} > \boldsymbol{e}^{\frac{1}{e}} \tag{177}$$

so the series $s(\boldsymbol{e}, z)_\infty$ converges at least where $|z| < \boldsymbol{e}^{\boldsymbol{e}^{-1}}$. Since $\boldsymbol{e}^{\boldsymbol{e}^{-1}} \approx 1.4447$, the required radius of convergence $R > 1$ is met, therefore the piecewise-definition of the superlogarithm in (9) is valid, and by the argument in (2.3.1), so is (10). □

4.4 Analytic Continuation of the Super-logarithm

The general solution for the infinite set of equations needed for $\mathrm{slog}_x(z)$ seems almost impossible, however, a general expression for the solutions to finite sets of equations needed for approximations can be achieved. Cramer's rule has been chosen to solve the matrix equations, and permutation enumeration has been chosen to evaluate determinants. The upper matrices used in Cramer's rule based on (44) are:

$$(C_m)_{jk} = A_{jk} - (A_{jk} - b_j)\delta_{mk} \tag{178}$$

where $1 \leq m \leq n$ and m is the column being replaced. This allows the solutions of the matrix equations to be written using Cramer's rule as:

$$u_k(x) = \frac{\det(\mathbf{C}_k)}{\det(\mathbf{A})} \tag{179}$$

$$= \frac{\sum_{p\in\mathrm{perm}_n} \epsilon_p \prod_{j=1}^{n} (C_k)_{jp_j}}{\sum_{p\in\mathrm{perm}_n} \epsilon_p \prod_{j=1}^{n} A_{jp_j}} \tag{180}$$

Combining this with (11) gives a single expression for $s(x,z)_n$ in terms of $\mathbf{A}$:

$$\boxed{s(x,z)_n = -1 + \sum_{k=1}^{n} z^k \frac{\sum_{p\in \mathrm{perm}_n} \epsilon_p \prod_{j=1}^{n} \left(A_{jp_j} - (A_{jp_j} - \delta_{(j-1)0})\delta_{kp_j}\right)}{\sum_{p\in \mathrm{perm}_n} \epsilon_p \prod_{j=1}^{n} A_{jp_j}}} \tag{181}$$

The piecewise error transform discussed in (2.1) allows the piecewise-defined $\mathrm{slog}_x(z)_n$ to be $\boldsymbol{C}^{(n-1)}$, so as $n \to \infty$, the superlogarithm $\mathrm{slog}_x(z)$ is $\boldsymbol{C}^{\infty}$. Thus, the definitions (9-10), (44) and (72) above form an infinitely-differentiable analytic continuation of the superlogarithm to the real numbers.

A Determinants

This table gives the determinants of the matrix $\mathbf{B}$ for small n where $w = \frac{1}{\log(x)}$:

Table 1: Determinants of $\mathbf{B}$

n	$\det(\mathbf{B})$
1	1
2	$1 + w$
3	$2 + 5w + 4w^2 + 2w^3$
4	$12 + 52w + 96w^2 + 110w^3 + 72w^4 + 36w^5 + 12w^6$
5	$288 + 1848w + 5664w^2 + 11140w^3 + 14760w^4 + 14688w^5$
	$+ 11340w^6 + 6432w^7 + 3024w^8 + 1152w^9 + 288w^{10}$

B References

[1] Euler, L. *De serie Lambertina Plurimisque eius insignibus proprietatibus.* Acta Acad. Scient. Petropol. 2, 29-51, 1783.

[2] Goodstein, R. L. *Transfinite ordinals in recursive number theory* Journal of Symbolic Logic 12, 1947.

[3] Mauerer, H. *Über die Funktion* $x^{x^{\cdot^{\cdot}}}$ *für ganzzahliges Argument (Abundanzen).* Mitt. Math. Gesell. Hamburg 4, 33-50, 1901.

[4] Rubtsov, C. A. and G. F. Romerio *Ackermann's Function and New Arithmetical Operations*, http://www.rotarysaluzzo.it/filePDF/Iperoperazioni%20(1).pdf, 2003.

[5] Rucker, R. *Infinity and the Mind: The Science and Philosophy of the Infinite.* Princeton, NJ: Princeton University Press, 1995.

[6] Stewart, J. *Single Variable Calculus: Concepts and Contexts*, Thomson Learning, 2005.