An Analytic Continuation of the Inverse of Iterated Exponentiation to the Reals

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Abstract

Specific differentiability and piecewise continuity requirements are applied to a piecewise-defined function representing the inverse of iterated exponentiation. An analytic continuation of the represented inverse results from solving a finite set of those requirements. Such a continuation then implies an analytic continuation of iterated exponentiation itself.

1 Introduction

Iterated exponentiation over integers is defined as:

$$ {}^{n}x = \underbrace{x^{x^{\cdot^{\cdot^{x}}}}}_{n}. \tag{1}$$

The study of iterated exponentiation began when Euler [1] (1783) showed the convergence of infinitely iterated exponentiation (${}^{\infty}x$) over a closed interval. Over a century later, Maurer [3] (1901) began using the notation ${}^{n}x$ for it, which was later popularized by Rucker [5] (1995). This *left-superscript* notation has become ubiquitous. For a shorter terminology, Goodstein [2] (1947) coined the term *tetration* to refer to iterated exponentiation. More recently, Rubtsov [4] (1989) uses the names *super-root*, and *super-logarithm* for the two inverses of tetration.

The super-logarithm is defined as: $\text{slog}_x(z) = y$ if and only if ${}^{y}x = z$. This effectively creates a sparse set of acceptable values for z (the range of tetration), since tetration is only defined for integer y. In order to extend the domains of tetration and the super-logarithm, a better understanding of their properties is needed.

1.1 Axioms of Tetration and Super-logarithms

The axiom of tetration is:

$$\boxed{{}^{y}x = x^{\left({}^{(y-1)}x\right)}}. \tag{2}$$

Given this property and a point (say, ${}^{0}x = 1$) one could define tetration over integers as in (1). It can also be used to find an equivalent axiom of super-logarithms.

Start with the definition of the super-logarithm, and replace y with $y + 1$:

$$y = \operatorname{slog}_x({}^{y}x), \tag{3}$$

$$y + 1 = \operatorname{slog}_x({}^{(y+1)}x), \tag{4}$$

$$y = \operatorname{slog}_x({}^{(y+1)}x) - 1. \tag{5}$$

Then expand (5) using (2), and equate to (3):

$$y = \operatorname{slog}_x(x^{({}^{y}x)}) - 1, \tag{6}$$

$$\operatorname{slog}_x({}^{y}x) = \operatorname{slog}_x(x^{({}^{y}x)}) - 1. \tag{7}$$

Replacing ${}^{y}x$ with z, the result is the axiom of super-logarithms:

$$\boxed{\operatorname{slog}_x(z) = \operatorname{slog}_x(x^z) - 1}. \tag{8}$$

1.2 Piecewise-Definition of Super-logarithms

Using (8) it is possible to piecewise-define super-logarithms in terms of $s(x, z)$:

$$\operatorname{slog}_x(z)_n = \begin{cases} s(x, x^z)_n - 1 & \text{if } z \le 0 \\ s(x, z)_n & \text{if } 0 < z \le 1 \\ s(x, \log_x^m(z))_n + m & \text{if } ({}^{(m-1)}x) < z \le ({}^{m}x), m \in \mathbb{Z}^+ \end{cases} \tag{9}$$

$$\operatorname{slog}_x(z) = \operatorname{slog}_x(z)_\infty \tag{10}$$

$$s(x, z)_n = \sum_{k=0}^{n} z^k u_k(x), \tag{11}$$

$$s(x, z) = s(x, z)_\infty. \tag{12}$$

The function $s(x, z)$ is still unknown, though, because the coefficients $u_k(x)$ are unknown. However, by requiring that the super-logarithm is finitely differentiable ($\boldsymbol{C}^n$) with respect to z, a set of equations can be constructed.

1.3 The Critical Function of the Super-logarithm

The function $s(x,z)$ can be called the *critical function* of the super-logarithm, because the piecewise-definition of $\text{slog}_x(z)$ forces all of the z values that get passed to $s(x,z)$ to be within the set (0, 1]. Since ${}^{-1}x = 0$ and ${}^{0}x = 1$, that means that $\text{slog}_x(0) = -1$ and $\text{slog}_x(1) = 0$, so the super-logarithm maps [0, 1] onto [-1, 0]. To produce proper values for the super-logarithm, its critical function must also obey $s(x,0) = -1$ and $s(x,1) = 0$. For a smooth and differentiable super-logarithm, the critical function should be monotonic in its mapping between [0, 1] and [-1, 0]. It is for this reason that the domain and range of $s(x,z)$ are restricted as such:

$$s : \mathbb{R}^+ \times [0,1] \mapsto [-1,0]. \tag{13}$$

2 Results

By requiring that $\text{slog}_x(z)$ is $\boldsymbol{C}^n$ with respect to z, a set of $n+1$ equations can be formed by analyzing the possible discontinuity at $z = 0$. In order to do this, a formal representation of the discontinuity between pieces is needed for each derivative.

2.1 The Piecewise Error Transform

The piecewise error transform is presented here to help construct the equations:

$$\boxed{\underset{z\to c}{\Delta^i}\{f(z)\} = \lim_{z\to c^-} \boldsymbol{D}^i f(z) - \lim_{z\to c^+} \boldsymbol{D}^i f(z)}, \tag{14}$$

which represents the magnitude of discontinuity between pieces of a piecewise-defined function, or its derivatives. To ensure that $\text{slog}_x(z)_n$ is $\boldsymbol{C}^{(n-1)}$ this must be zero:

$$\boxed{\underset{z\to 0}{\Delta^i}\{\text{slog}_x(z)_n\} = 0 \text{ for all } 0 \le i < n}. \tag{15}$$

Note that this represents n equations, and that $\text{slog}_x(z)_n$ is defined in terms of $s(x,z)_n$ which contains n unknowns, so its possible that this system of n equations in n unknowns contains at least one, possibly unique, solution.

Remember that ${}^{-1}x = 0$, which implies $\text{slog}_x(0) = -1$ for all x, which implies $s(x,0) = -1$, so it follows that $u_0 = -1$. Also, for the sake of simplicity, unknowns will not be written as functions (i.e. $u_k = u_k(x)$).

2.2 Piecewise Error Transform of the Super-logarithm

From the definition of the piecewise error transform and the definition of the super-logarithm presented here, a general form of the previous equations can be shown:

$$\begin{aligned}
\underset{z\to 0}{\Delta^i}\{\mathrm{slog}_x(z)_n\} &= \boldsymbol{D}_z^i[s(x,x^z)_n - 1]_{z=0} && - \boldsymbol{D}_z^i[s(x,z)_n]_{z=0}, && (16)\\
&= \boldsymbol{D}_z^i\left[\sum_{k=0}^{n} x^{zk}u_k(x) - 1\right]_{z=0} && - \boldsymbol{D}_z^i\left[\sum_{k=0}^{n} z^k u_k(x)\right]_{z=0} && (17)\\
&= \boldsymbol{D}_z^i\left[\sum_{k=1}^{n} x^{zk}u_k(x) - 1\right]_{z=0} && - \boldsymbol{D}_z^i\left[\sum_{k=1}^{n} z^k u_k(x)\right]_{z=0} && (18)\\
&= \left[\sum_{k=1}^{n} (k\log(x))^i x^{zk}u_k(x) - \delta_{i0}\right]_{z=0} && - \left[\sum_{k=i}^{n} i! z^{k-i}u_k(x)\right]_{z=0} && (19)\\
&= \sum_{k=1}^{n} k^i \log(x)^i u_k(x) - \delta_{i0} && - i! u_i(x), && (20)\\
&= \sum_{k=1}^{n}(k^i\log(x)^i - k!\delta_{ik})u_k(x) - \delta_{i0}, && && (21)
\end{aligned}$$

where $0 \leq i < n$, δ_{jk} is the Kronecker delta, and in equation (41) only: $0! = 0$.

The equations in (15) can now be rewritten as:

$$\sum_{k=1}^{n}(k^i\log(x)^i - k!\delta_{ik})u_k(x) = \delta_{i0} \text{ for all } 0 \leq i < n. \tag{22}$$

Notice that all the equations are linear, and are thus expressible in matrix form. Replace i with $j-1$ in (41) where $1 \leq j \leq n$, and a general expression for the matrix form of the equations is $\mathbf{Au} = \mathbf{b}$, where:

$$\boxed{A_{jk} = k^{(j-1)}\log(x)^{(j-1)} - k!\delta_{(j-1)k}}, \tag{23}$$

$$u_k = u_k(x), \tag{24}$$

$$\boxed{b_j = \delta_{(j-1)0}}. \tag{25}$$

In order to make the matrices more clear, a variant of the above matrix is shown, $\mathbf{B}$ where $B_{jk} = A_{jk}/\log(x)^{(j-1)}$, so that the equations still hold, and the majority of components are integers. This illustrates the set of equations for $\text{slog}_x(z)_3$:

$$\begin{pmatrix} 1 & 1 & 1 \\ 1-\frac{1}{\log(x)} & 2 & 3 \\ 1 & 4-\frac{2}{\log(x)^2} & 9 \end{pmatrix} \mathbf{u} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \tag{26}$$

2.3 Existence, Uniqueness and Convergence

To determine the existence and uniqueness of solutions to the above equations, their determinants will be found. Determinants will be evaluated by enumerating all of the index permutations. As a reminder, the permutation enumeration method for evaluating the determinant of a square $n \times n$ matrix is defined as:

$$\det(\mathbf{A}) = \sum_{p \in \text{perm}_n} \epsilon_p \prod_{j=1}^{n} A_{jp_j} \tag{27}$$

where ϵ_p is the permutation symbol, $\text{perm}_n = \text{perm}(N_1^n)$, $\text{perm}(S)$ is the set of all permutations of the elements of S, and $N_m^n = \{m, (m+1), ..., (n-1), n\}$.

Using the expressions for $\mathbf{A}$ and $\mathbf{B}$, their determinants in general are:

$$\det(\mathbf{A}) = \sum_{p \in \text{perm}_n} \epsilon_p \prod_{j=1}^{n} ((p_j)^{(j-1)} \log(x)^{(j-1)} - (p_j)!\delta_{(j-1)p_j}) \tag{28}$$

$$\det(\mathbf{B}) = \sum_{p \in \text{perm}_n} \epsilon_p \prod_{j=1}^{n} \left((p_j)^{(j-1)} - \frac{(p_j)!\delta_{(j-1)p_j}}{\log(x)^{(j-1)}} \right) \tag{29}$$

$$= \sum_{a \in L} \left(\prod_{k \in a} \frac{k!}{\log(x)^k} \right) \sum_{b \in Q} \epsilon_b \prod_{\substack{c \in R \\ d \in b}} c^d \tag{30}$$

$$= \sum_{k=0}^{n(n-1)/2} \frac{g_k}{\log(x)^k} \tag{31}$$

where L the set of all subsets of N_1^n, $Q = \text{perm}(N_0^{n-1} - \{a\})$, and $R = N_1^n - \{a\}$.

All g_k are positive (see appendix A), which implies: $\det(\mathbf{B}) = 0$ which implies $\log(x) < 0$ which implies $x < 1$, the negation of which would be:

$$x > 1 \implies \det(\mathbf{B}) \neq 0. \tag{32}$$

Since a unique solution exists for $\mathbf{Bu} = \mathbf{b}$ only when $\det(\mathbf{B})$ is nonzero, one exists when $x > 1$. By restricting x to be greater than one, there exists exactly one solution to the set of equations $\mathbf{B}$ represents. Since $\mathbf{B}$ is row-equivalent to $\mathbf{A}$, it too is invertible, so there is one and only one solution to the equations $\mathbf{A}$ represents. $\square$

2.3.1 Convergence of Approximants to the Super-logarithm Series

To show convergence of the approximants $s(x, z)_n$ to the infinite series $s(x, z)_\infty$, an example proof for $x = \boldsymbol{e}$ will be shown. The approximants will converge if the absolute error approaches zero as $n \to \infty$. Since absolute error requires an accurate value to compare to, we will use the $(2n)$th approximant in place of a perfect value. This leads to the definition of absolute error between approximants as such:

$$E_A = |s(x, z)_n - s(x, z)_{2n}| \tag{33}$$

An upper bound for the absolute error when $x = \boldsymbol{e}$ is:

$$\boxed{|s(\boldsymbol{e}, z)_n - s(\boldsymbol{e}, z)_{2n}| < (n+2)^{-\frac{7}{2}}} \text{ if } z \in [0, 1], \tag{34}$$

$$\lim_{n \to \infty} (n+2)^{-\frac{7}{2}} = 0, \tag{35}$$

so the approximants $s(\boldsymbol{e}, z)_n$ converge to the series $s(\boldsymbol{e}, z)_\infty$ as $n \to \infty$. $\square$

2.3.2 Convergence of Coefficients of the Super-logarithm Series

To show that the super-logarithm itself converges (or $s(x, z)_\infty$), a better look at the coefficients is needed. For $n > 7$ a pattern emerges in the coefficients: the signs of any seven coefficients are opposite in the next seven coefficients. Remembering $u_0 = -1$, the signs of the coefficients $\{u_0, u_1, ..., u_{13}\}$ for $x = \boldsymbol{e}$ are:

$$\begin{aligned} \{&-, +, +, -, -, +, +, \\ &+, -, -, +, +, -, -\} \end{aligned} \tag{36}$$

which allows the ratio between every seven coefficients to be approximated as:

$$\frac{u_{k+7}(\boldsymbol{e})}{u_k(\boldsymbol{e})} \approx -k^{7(\frac{7}{k})^{\frac{1}{k}}} \tag{37}$$

An upper bound for the coefficients $u_k(\boldsymbol{e})$ is:

$$\boxed{|u_k(\boldsymbol{e})| < \boldsymbol{e}^{-k\left(\frac{1}{e}+\frac{1}{7k}\right)}} \text{ if } k > 2, \tag{38}$$

$$\lim_{k \to \infty} \boldsymbol{e}^{-k\left(\frac{1}{e}+\frac{1}{7k}\right)} = 0, \tag{39}$$

and since the series $s(\boldsymbol{e}, z)_\infty$ has alternating signs every seven coefficients, by the *alternating series test* [6], the series $s(\boldsymbol{e}, z)_\infty$ converges over some interval. $\square$

2.3.3 Radius of Convergence of the Super-logarithm Series

To find the radius of convergence of the series $s(\boldsymbol{e}, z)_\infty$, the limit of the supremum of the roots of the coefficients is needed. An upper bound of the roots is:

$$\boxed{\left|u_k(\boldsymbol{e})^{\frac{1}{k}}\right| < \boldsymbol{e}^{-\left(\frac{1}{e}+\frac{1}{7k}\right)}} \text{ if } k > 2, \tag{40}$$

which implies:

$$c = \lim_{n\to\infty} \sup_{k\geq n} \left|u_k(\boldsymbol{e})^{\frac{1}{k}}\right| < \lim_{n\to\infty} \boldsymbol{e}^{-\left(\frac{1}{e}+\frac{1}{7n}\right)} = \boldsymbol{e}^{-\frac{1}{e}} \tag{41}$$

$$R = \frac{1}{c} > \boldsymbol{e}^{\frac{1}{e}} \tag{42}$$

so the series $s(\boldsymbol{e}, z)_\infty$ converges at least where $|z| < \boldsymbol{e}^{\boldsymbol{e}^{-1}}$. Since $\boldsymbol{e}^{\boldsymbol{e}^{-1}} \approx 1.4447$, the required radius of convergence $R > 1$ is met, therefore the piecewise-definition of the super-logarithm in (9) is valid, and by the argument in (2.3.1), so is (10). $\square$

2.4 Analytic Continuation of the Super-logarithm

The general solution for the infinite set of equations needed for $\mathrm{slog}_x(z)$ seems almost impossible, however, a general expression for the solutions to finite sets of equations needed for approximations can be achieved. Cramer's rule has been chosen to solve the matrix equations, and permutation enumeration has been chosen to evaluate determinants. The upper matrices used in Cramer's rule based on (44) are:

$$(C_m)_{jk} = A_{jk} - (A_{jk} - b_j)\delta_{mk} \tag{43}$$

where $1 \leq m \leq n$ and m is the column being replaced. This allows the solutions of the matrix equations to be written using Cramer's rule as:

$$u_k(x) = \frac{\det(\mathbf{C}_k)}{\det(\mathbf{A})} \tag{44}$$

$$= \frac{\sum_{p\in\mathrm{perm}_n} \epsilon_p \prod_{j=1}^n (C_k)_{jp_j}}{\sum_{p\in\mathrm{perm}_n} \epsilon_p \prod_{j=1}^n A_{jp_j}} \tag{45}$$

Combining this with (11) gives a single expression for $s(x, z)_n$ in terms of $\mathbf{A}$:

$$\boxed{s(x, z)_n = -1 + \sum_{k=1}^n z^k \frac{\sum_{p\in\mathrm{perm}_n} \epsilon_p \prod_{j=1}^n (A_{jp_j} - (A_{jp_j} - \delta_{(j-1)0})\delta_{kp_j})}{\sum_{p\in\mathrm{perm}_n} \epsilon_p \prod_{j=1}^n A_{jp_j}}} \tag{46}$$

The piecewise error transform discussed in (2.1) allows the piecewise-defined $\mathrm{slog}_x(z)_n$ to be $\boldsymbol{C}^{(n-1)}$, so as $n \to \infty$, the super-logarithm $\mathrm{slog}_x(z)$ is $\boldsymbol{C}^{\infty}$. Thus, the definitions (9-10), (44) and (72) above form an infinitely-differentiable analytic continuation of the super-logarithm to the real numbers.

A Determinants

This table gives the determinants of the matrix $\mathbf{B}$ for small n where $w = \frac{1}{\log(x)}$:

Table 1: Determinants of $\mathbf{B}$

n	$\det(\mathbf{B})$
1	1
2	$1 + w$
3	$2 + 5w + 4w^2 + 2w^3$
4	$12 + 52w + 96w^2 + 110w^3 + 72w^4 + 36w^5 + 12w^6$
5	$288 + 1848w + 5664w^2 + 11140w^3 + 14760w^4 + 14688w^5$
	$+ 11340w^6 + 6432w^7 + 3024w^8 + 1152w^9 + 288w^{10}$

B References

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