A *Fee* Subgroup Is Not Necessarily Normal

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Abstract

For any positive element g of G, $g^{-1}Hg \subset H$ holds, but H is not normal; that is, there exists a negative element g_1 such that $g_1^{-1}Hg_1 \not\subset H$. We construct an example of such a subgroup H of a totally ordered group G. If we bi-order the free group F generated by $\{a, b\}$ by the method of Magnus ordering, then $H = \langle b^{F_+} \rangle$ and $g_1 = a^{-1}$ give an example satisfying the above condition. As a lemma for this, let F be the free group generated by X, $x \in X$, and $U \subset x^F$; we show that there exists a subset of x^F which freely generates $\langle U \rangle$.

This article is a translation of my blog post ”フィーは必ずしも正規でない” [8]. I used ChatGPT to translate the original sentence except the proof part. I checked all the results and corrected any mistakes made by the AI.

0 Relation with Hyperoperations

Hyperoperations satisfy the identity $m[r+1](n+1) = m[r](m[r+1]n)$ for any natural numbers m, n and positive r. If we set $a_n = m[r+1]n$ and $f(n) = m[r]n$, then this becomes the recurrence relation $a_{n+1} = f(a_n)$. Therefore, in order to extend the sequence of hyperoperations, it is necessary to study the properties of this recurrence.

By repeatedly applying $a_{n+1} = f(a_n)$, one finds that $a_{m+n} = f^m(a_n)$ for any positive integer m. This can equivalently be stated as that $a_m = a_n \implies a_{m+k} = a_{n+k}$ for any positive integer k.

Moreover, for some a_n, if there exists a k such that $f^k(a_n) = a_n$, then we call k a period. The set of periods $P(a)$, which for convenience we allow to include 0 and disregard the distinction between positive and negative, is given by

$$\begin{aligned}
P(a) &= \{k \mid k > 0, \exists n \in \mathbb{Z}, f^k(a_n) = a_n\} \cup \{0\} \cup \{-k \mid k > 0, \exists n \in \mathbb{Z}, f^k(a_n) = a_n\} && (1)\\
&= \{k \mid k > 0, \exists n \in \mathbb{Z}, a_{n+k} = a_n\} \cup \{0\} \cup \{-k \mid k > 0, \exists n \in \mathbb{Z}, a_{n+k} = a_n\} && (2)\\
&= \{m - n \mid m - n > 0, a_{n+m-n} = a_n\} \cup \{0\} \cup \{-(m - n) \mid m - n > 0, a_{n+m-n} = a_n\} && (3)\\
&= \{m - n \mid m > n, a_m = a_n\} \cup \{0\} \cup \{n - m \mid n < m, a_n = a_m\} && (4)\\
&= \{m - n \mid a_m = a_n\}. && (5)
\end{aligned}$$

The periods $P(a)$ is a subgroup of $\mathbb{Z}$.

By extending the domain of a from the integers to the real numbers, and requiring that $a(x) = a(y) \implies a(x + z) = a(y + z)$ for any $z > 0$, and defining $P(a) = \{x - y \mid x, y \in$

$\mathbb{R}, a(x) = a(y)\}$, we obtain a real-number version of the recurrence-like structure. It is easy to see that $P(a)$ is a subgroup of $\mathbb{R}$.

This can also be interpreted as follows: if we regard the variable of a as time, then starting from the same state, the system reaches the same state after the same amount of elapsed time.

There are cases where a has infinitely small periods, such as when $a : \mathbb{R} \to \mathbb{R}/\mathbb{Q}, \quad x \mapsto x+\mathbb{Q}$. In this case, $P(a) = \mathbb{Q}$.

We now consider extending the domain of a further to a totally ordered group G. Since a totally ordered group is not necessarily commutative, we require that $a(g) = a(h) \implies a(fg) = a(fh)$ and $a(gf) = a(hf)$ for any $f > 1$, and define $P(a) = \{g^{-1}h \mid g, h \in G, a(g) = a(h)\}$.

In this setting, if G is a BO-group, then $P(a)$ is a subgroup of G. Moreover, it can be shown that $g^{-1}P(a)g \subset P(a)$ holds for any $g > 1$. In commutative cases such as $\mathbb{Z}$ or $\mathbb{R}$, the above property is evident without the need for discussion. This is the content of the previous article [1].

The property of $P(a)$ stated above is weaker than the condition that a subgroup H is normal, namely, that $g^{-1}Hg \subset H$ for every $g \in G$ (including both positive and negative elements). The question we now address is whether situations can exist in which the former condition holds while the latter does not.

1 A *Fee* Subgroup Is Not Necessarily Normal

Definition 1.1. Let G be an LO-group and $H \leq G$. If for every positive element g of G we have $g^{-1}Hg \subset H$, then H is said to be *fee*, and we write $H \trianglelefteq^+ G$.

We would like to show that a *fee* subgroup is not necessarily normal. Since any subgroup of a commutative group is normal, we first need to construct a non-commutative LO-group.

Let F be the free group generated by $\{a, b\}$. Of course, F is non-commutative. We bi-order F by the method of [4, Definition 3]. For subsets $S, T \subset F$, we write $T^S = \{t^s \mid s \in S, t \in T\} = \{s^{-1}ts \mid s \in S, t \in T\}$. We also write t^S instead of $\{t\}^S$. For a totally ordered group G, let $G_+ = \{g \in G \mid g > 1\}$. The subgroup $\langle b^{F_+} \rangle$ is the smallest *fee* containing b [1], but it seems not to contain aba^{-1}. If that is the case, then $\langle b^{F_+} \rangle$ is not normal. To show this, we need to examine what kind of set freely generates $\langle b^{F_+} \rangle$.

The following content is a summary of [5, pp. 4 – 8]. Let F be a free group. For $U \subset F$, define $U^{-1} = \{u^{-1} \mid u \in U\}$. From now on, we identify U with the vector listing all of its elements, $(u_1, u_2, \dots)$. Moreover, we do not distinguish between U and $U \cup U^{-1}$, and we assume that if $u \in U$ then $u^{-1} \in U$ as well. A subset $U \subset F$ is said to be N-reduced if, for any $v_1, v_2, v_3 \in U$, the following three conditions hold:

(N0) $v_1 \neq 1$;

(N1) $v_1v_2 \neq 1$ implies $|v_1v_2| \geq |v_1|, |v_2|$;

(N2) $v_1v_2 \neq 1$ and $v_2v_3 \neq 1$ implies $|v_1v_2v_3| > |v_1| - |v_2| + |v_3|$.

A subset U freely generates $\langle U \rangle$ (that is, it satisfies only the trivial relations $u_iu_i^{-1} = 1$) if and only if for $n \geq 1$, if for each i we have $u_i \in U$ and $u_iu_{i+1} \neq 1$, then $u_1 \cdots u_n \neq 1$. If U is N-reduced, then in the product $u_1 \cdots u_n$ none of the elements u_i are completely canceled, so we have $|u_1 \cdots u_n| \geq n \geq 1$, and hence $u_1 \cdots u_n \neq 1$.

A Nielsen transformation for $U = (u_1, u_2, \dots) \subset F$ is defined as the composition of the following three types of transformations:

(T1) replace some u_i by u_i^{-1};

(T2) replace some u_i by $u_i u_j$ where $j \neq i$;

(T3) delete some u_i where $u_i = 1$.

Replacing u_i with $u_j u_i$ can be achieved as the composition of (T1) and (T2). When U is finite, by applying these three types of transformations finitely many times, we can obtain an N-reduced set V such that $\langle V \rangle = \langle U \rangle$. By applying (T2) whenever $|u_i u_j| < |u_i|$, and continuing this process until no element can be further shortened, we can make U N-reduced. A similar argument also applies when U is infinite.

The above is a summary of [5, pp. 4 – 8]. Now, when performing a Nielsen transformation on $b^{F_+} = (s_1^{-1} b^{e_1} s_1, s_2^{-1} b^{e_2} s_2, \dots)$, if $|s_i^{-1} b^{e_i} s_i s_j^{-1} b^{e_j} s_j| < |s_i^{-1} b^{e_i} s_i|$, then since $s_i^{-1} b^{e_i} s_i$ is symmetric in form, we may perform the same operation once more on the left to obtain $s_j^{-1} b^{-e_j} s_j s_i^{-1} b^{e_i} s_i s_j^{-1} b^{e_j} s_j$ whose length appears to be even shorter than that of $s_i^{-1} b^{e_i} s_i s_j^{-1} b^{e_j} s_j$. That is, whenever we perform the replacement $u_i \to u_i u_j$, we always follow it by $u_i u_j \to u_j^{-1} u_i u_j$. Hence, by modifying (T2), we consider only the following three types of transformations:

(T1) replace some u_i by u_i^{-1};

(T2$^+$) replace some u_i by $u_j^{-1} u_i u_j$ where $j \neq i$;

(T3) delete some u_i where $u_i = 1$.

Using only this combination of transformations, b^{F_+} can be transformed into an N-reduced set V, and in that case, V becomes a subset of b^F. This is the idea of the present discussion.

Although this is not necessary for proving the main claim, we shall first write down the proof that a Nielsen transformation can convert U into a set V which freely generates $\langle U \rangle$, given in [6] for the finite case of U and in [7] for the infinite case. This will make it easier to understand the argument by comparison.

For a free group F generated by a set of letters X, we define an order $\preceq$ on $\{\{w, w^{-1}\} \mid w \in F\}$ as the following. For $w \in F$ we define $L(w)$ as the left half of the reduced form of w, where it contains the central letter if $|w|$ is odd. Also, let $X \cup \{a^{-1} \mid a \in X\}$ has a lexicographical order. We define $\{w_1, w_1^{-1}\} \preceq \{w_2, w_2^{-1}\}$ if and only if: $|w_1| < |w_2|$, or else $|w_1| = |w_2|$ and $\min\{L(w_1), L(w_1^{-1})\} < \min\{L(w_2), L(w_2^{-1})\}$ on the lexicographical order, otherwise both are equal and $\max\{L(w_1), L(w_1^{-1})\} < \max\{L(w_2), L(w_2^{-1})\}$, if not so both are also equal, that is $w_1 = w_2$. In addition , we write $w_1 \preceq w_2$ when $\{w_1, w_1^{-1}\} \preceq \{w_2, w_2^{-1}\}$. Under this definition, for $U \subset F$ and $u \in \langle U \rangle$ we define

$$\alpha(u, U) = \langle \{v \in U \mid v \prec u\} \rangle, A(U) = \{u \in \langle U \rangle \mid u \notin \alpha(u, U)\}. \tag{6}$$

Then the followings hold.

$|w_1| < |w_2| \implies w_1 \prec w_2$: It's obvious from the definition.

$\prec$ is a well-order: Let U be a non-empty subset of F. Firstly, since $\mathbb{N}$ is well-ordered, $n = \min\{|u| \mid u \in U\}$ exists. Then $\{u \in U \mid |u| = n\}$ has a least element as it's finite. The element is also the least element of U because $|w_1| < |w_2| \implies w_1 \prec w_2$.

$\langle A(U) \rangle = \langle U \rangle$: From the definition $A(U) \subset \langle U \rangle$, so $\langle A(U) \rangle \subset \langle U \rangle$. Assume $\langle A(U) \rangle \subsetneq \langle U \rangle$. Then $\langle U \rangle - \langle A(U) \rangle$ is not empty. It has a least element u because $\prec$ is a well-order. As u is least, any $v \prec u$ is in $\langle A(U) \rangle$, and so products of elements less than u are also in $\langle A(U) \rangle$ which does not contain u. That indicates $u \notin \alpha(u,U)$. From the definition we get $u \in A(U) \subset \langle A(U) \rangle$, that is a contradiction.

(N0): Since $|w_1| < |w_2| \implies w_1 \prec w_2$, 1 is the minimum element of $(F, \preceq)$. Thus $1 \in \alpha(1,U) = \langle \{v \in U \mid v \prec 1\} \rangle = \langle \varnothing \rangle = \{1\}$. Therefore $1 \notin A(U)$.

(N1): When $w = u_1 \dots u_n$ and $|w| = |u_1| + \cdots + |u_n|$ holds, we say $w = u_1 \dots u_n$ holds without cancellation, or $w = u_1 \dots u_n$ reduced (on the right side).

Firstly we show $|u^2| < |u|$ is impossible in a free group. Let p denotes the cancellation in the product uu; there exist a, b and p such that $u = ap^{-1}$, $u = pb$ and $uu = ab$ hold without cancellation. Suppose $|u^2| < |u|$, that is $|a| < |p|$. Since $ap^{-1} = pb$ reduced on both sides, the first $|a|$ letters of p is a. Also, the the last $|b| = |a|$ letters of p^{-1} is b. Therefore $b = a^{-1}$ and so $u^2 = ab = 1$. This contradicts F being free group.

Assume that there exist $v_1, v_2 \in A(U)$ such that $v_1 v_2 \neq 1$ and $|v_1 v_2| < |v_1|, |v_2|$. By the above we see $v_1 \neq v_2$. We can assume that $v_1 \prec v_2$ without loss of generality. Then $v_1^{-1} \prec v_2$, $v_1 v_2 \prec v_2$ and $v_1^{-1} v_1 v_2 = v_2$ holds. That leads $v_2 \in \alpha(v_2, U)$, contradicting the definition of $A(U)$.

(N2): Assume that there exist $v_1, v_2, v_3 \in A(U)$ such that $v_1 v_2 \neq 1$, $v_2 v_3 \neq 1$ and $|v_1 v_2 v_3| \leq |v_1| - |v_2| + |v_3|$. As we have checked $A(U)$ satisfies (N1), no more than half of the reduced word v_2 can be canceled. Therefore we can write each cancellation as p, q so that $v_1 = ap^{-1}$, $v_2 = pbq^{-1}$ (if $b = 1$, pq^{-1}), $v_3 = qc$ and $v_1 v_2 v_3 = abc$ (if $b = 1$, ac) , all reduced.

(i) Where $b \neq 1$: Since there is no cancellation in ab and bc, we get $|v_1 v_2 v_3| = |a| + |b| + |c| > |a| - |b| + |c| = (|a| + |p|) - (|p| + |b| + |q|) + (|q| + |c|) = |v_1| - |v_2| + |v_3|$, a contradiction.

(ii) Where $b = 1$: The cancellations p, q do not occupy more than half of v_2, v_1 and v_3 because $A(U)$ satisfies (N1). Thus $|v_2| = |p| + |q| = 2|p|$, $|v_1| \geq 2|p| = |v_2|$ and $|v_3| \geq 2|q| = |v_2|$. We see $p \neq q$ because $p = q$ implies $v_2 = pq^{-1} = pp^{-1} = 1$, contradicting (N0).

(ii-i) Where $p < q$ on the lexicographical order: Since $L(v_2) = L(pq^{-1}) = p < q = L(qp^{-1}) = L(v_2^{-1})$, $\min\{L(v_2), L(v_2^{-1})\} = L(v_2)$.

(ii-i-i) Where $\min\{L(v_3), L(v_3^{-1})\} = L(v_3)$: From $|v_3| \geq |v_2|$, $|v_3| > |v_2|$ or $|v_3| = |v_2|$. If the former holds, $v_2 \prec v_3$. If the latter holds, since $p < q$, $\min\{L(v_2), L(v_2^{-1})\} = L(v_2) = L(pq^{-1}) = p < q = L(qc) = L(v_3) = \min\{L(v_3), L(v_3^{-1})\}$, and so $v_2 \prec v_3$. Therefore always $v_2 \prec v_3$. Furthermore, since $|v_2 v_3| = |pc| = |qc| = |v_3|$ and $L(v_2 v_3) = p \cdots < q \cdots = L(v_3) \leq L(v_3^{-1}) = L(c^{-1} q^{-1}) = L(c^{-1} p^{-1}) = L((v_2 v_3)^{-1})$, we get $\min\{L(v_2 v_3), L(v_2 v_3^{-1})\} = L(v_2 v_3) = p \cdots < q \cdots = \min\{L(v_3), L(v_3^{-1})\}$, that is $v_2 v_3 \prec v_3$. Thus $v_2^{-1} \prec v_3$, $v_2 v_3 \prec v_3$ and $v_2^{-1} v_2 v_3 = v_3$ holds. That leads $v_3 \in \alpha(v_3, U)$, contradicting the definition of $A(U)$.

(ii-i-ii) Where $\min\{L(v_3), L(v_3^{-1})\} = L(v_3^{-1})$:

(ii-i-ii-i) Where $L(v_2 v_3) < L(v_3^{-1})$: We know that $|v_3| > |v_2|$ or $|v_3| = |v_2|$. If the former holds, $v_2 \prec v_3$. If the latter holds, $\min\{L(v_2), L(v_2^{-1})\} = L(v_2) = L(pq^{-1}) = p = L(pc) = L(v_2 v_3) < L(v_3^{-1}) = \min\{L(v_3), L(v_3^{-1})\}$, and so $v_2 \prec v_3$. Therefore always $v_2 \prec v_3$. We also know $L(v_2 v_3) < L(v_3^{-1}) = \min\{L(v_3), L(v_3^{-1})\}$. If $L((v_2 v_3)^{-1}) \geq \min\{L(v_3), L(v_3^{-1})\}$, $\min\{L(v_2 v_3), L(v_2 v_3)^{-1}\} = L(v_2 v_3) < \min\{L(v_3), L(v_3^{-1})\}$. If $L((v_2 v_3)^{-1}) <$ $\min\{L(v_3), L(v_3^{-1})\}$, by considering that $\min\{L(v_2 v_3), L(v_2 v_3)^{-1}\}$ is $L(v_2 v_3)$ or $L(v_2 v_3)^{-1}$, we see $\min\{L(v_2 v_3), L(v_2 v_3)^{-1}\} < \min\{L(v_3), L(v_3^{-1})\}$. We therefore get always $v_2 v_3 \prec v_3$. Thus $v_2^{-1} \prec v_3$, $v_2 v_3 \prec v_3$ and $v_2^{-1} v_2 v_3 = v_3$ holds. That leads $v_3 \in \alpha(v_3, U)$, contradicting the definition of $A(U)$.

(ii-i-ii-ii) Where $L(v_2 v_3) \geq L(v_3^{-1})$: As $L((v_2 v_3)^{-1}) = L(c^{-1} p^{-1}) = L(c^{-1} q^{-1}) = L(v_3^{-1}) \leq L(v_2 v_3)$, we see $\min\{L(v_2 v_3), L((v_2 v_3)^{-1})\} = \min\{L(v_3), L(v_3^{-1})\}$.

(ii-i-ii-ii-i) Where $|v_2| < |v_3|$: This implies $v_2 \prec v_3$. As $p < q$, $\max\{L(v_2 v_3), L((v_2 v_3)^{-1})\} =$

$L(v_2v_3) = L(pc) = p\cdots < q\cdots = L(qc) = L(v_3) = \max\{L(v_3), L(v_3^{-1})\}$, and so $v_2v_3 \prec v_3$. That leads $v_3 \in \alpha(v_3, U)$, a contradiction.

(ii-i-ii-ii-ii) Where $|v_2| = |v_3|$: This implies $\min\{L(v_3), L(v_3^{-1})\} = L(v_3^{-1}) \leq L(v_2v_3) = L(pc) = p = L(pq^{-1}) = L(v_2) = \min\{L(v_2), L(v_2^{-1})\}$. Assume $L(v_3^{-1}) = L(v_2)$, then $c^{-1} = L(c^{-1}q^{-1}) = L(v_3^{-1}) = L(v_2) = L(pq^{-1}) = p$, and hence $v_3^{-1} = c^{-1}q^{-1} = pq^{-1} = v_2$, contradicting $v_2v_3 \neq 1$. Therefore $L(v_3^{-1}) < L(v_2)$, so $v_3 \prec v_2$. Additionally, since $c^{-1} = L(v_3^{-1}) < L(v_2) = p$, $\min\{L(v_2v_3), L((v_2v_3)^{-1})\} = \min\{p, c^{-1}\} = c^{-1} < p = L(v_2) = \min\{L(v_2), L(v_2^{-1})\}$, that is $v_2v_3 \prec v_2$. That leads $v_2 \in \alpha(v_2, U)$, a contradiction.

(ii-ii) Where $q < p$: After substituting $(v_1, v_2, v_3) = (v_3^{-1}, v_2^{-1}, v_1^{-1})$, the initial assumption still holds, and a, b, c, p, q are replaced by $c^{-1}, b^{-1}, a^{-1}, q, p$, respectively. We can reach a contradiction by repeating same argument as in (ii-i).

Lemma 1.2. *For $U \subset F$, we define*

$$\begin{aligned} \overset{-}{\langle} U \overset{+}{\rangle} &= \{(u_1^{e_1})^{\cdot^{\cdot^{\cdot^{(u_n^{e_n})}}}} \mid n \in \mathbb{N}, u_1, \ldots, u_n \in U, e_1, \ldots, e_n = \pm 1\} && (7)\\ &= \{u_n^{-e_n} \cdots u_2^{-e_2} u_1^{e_1} u_2^{e_2} \cdots u_n^{e_n} \mid n \in \mathbb{N}, u_1, u_2, \ldots u_n \in U, e_1, e_2, \ldots, e_n = \pm 1\}, && (8) \end{aligned}$$

where $1 \in \overset{-}{\langle} U \overset{+}{\rangle}$ for any U. Further, for $U \subset F$ and $u \in \langle U \rangle$ we define

$$\beta(u, U) = \overset{-}{\langle}\{v \in U \mid v \prec u\}\overset{+}{\rangle}, B(U) = \{u \in \overset{-}{\langle} U \overset{+}{\rangle} \mid u \notin \beta(u, U)\} \qquad (9)$$

Let F be a free group generated by X, $x \in X$ and $S \subset F$. Then $B(x^S)$ freely generates $\langle x^S \rangle$.

Proof. $\langle B(x^S)\rangle = \langle x^S \rangle$: Firstly we show $\overset{-}{\langle} B(x^S) \overset{+}{\rangle} = \overset{-}{\langle} x^S \overset{+}{\rangle}$. By the definition $B(x^S) \subset \overset{-}{\langle} x^S \overset{+}{\rangle}$, so $\overset{-}{\langle} B(x^S) \overset{+}{\rangle} \subset \overset{-}{\langle}\overset{-}{\langle} x^S \overset{+}{\rangle}\overset{+}{\rangle} = \overset{-}{\langle} x^S \overset{+}{\rangle}$. Assume $\overset{-}{\langle} B(x^S) \overset{+}{\rangle} \subsetneq \overset{-}{\langle} x^S \overset{+}{\rangle}$. Then $\overset{-}{\langle} x^S \overset{+}{\rangle} - \overset{-}{\langle} B(x^S) \overset{+}{\rangle}$ is not empty. It has a least element u because $\prec$ is a well-order. As u is least, $\{v \in U \mid v \prec u\} \subset \overset{-}{\langle} B(x^S) \overset{+}{\rangle}$, and so $\beta(u, x^S) = \overset{-}{\langle}\{v \in U \mid v \prec u\}\overset{+}{\rangle} \subset \overset{-}{\langle}\overset{-}{\langle} B(x^S) \overset{+}{\rangle}\overset{+}{\rangle} = \overset{-}{\langle} B(x^S) \overset{+}{\rangle} \not\ni u$. That indicates $u \notin \beta(u, U)$. From the definition we get $u \in B(x^S) \subset \overset{-}{\langle} B(x^S) \overset{+}{\rangle}$, that is a contradiction.

Secondly we show $\langle \overset{-}{\langle} U \overset{+}{\rangle} \rangle = \langle U \rangle$ for any $U \subset F$. For all $u \in U$, by substituting $n = 1, u_1 = u, e_1 = 1$, we get $(u_1^{e_1})^{\cdot^{\cdot^{\cdot^{(u_n^{e_n})}}}} = u$, and so $U \subset \overset{-}{\langle} U \overset{+}{\rangle}$. Thus $\langle U \rangle \subset \langle \overset{-}{\langle} U \overset{+}{\rangle} \rangle$. Since $\overset{-}{\langle} U \overset{+}{\rangle} \subset \langle U \rangle$, $\langle \overset{-}{\langle} U \overset{+}{\rangle} \rangle \subset \langle \langle U \rangle \rangle = \langle U \rangle$. Therefore $\langle \overset{-}{\langle} U \overset{+}{\rangle} \rangle = \langle U \rangle$.

From the above $\langle B(x^S) \rangle = \langle \overset{-}{\langle} B(x^S) \overset{+}{\rangle} \rangle = \langle \overset{-}{\langle} x^S \overset{+}{\rangle} \rangle = \langle x^S \rangle$

(N0): Since $|w_1| < |w_2| \implies w_1 \prec w_2$, 1 is the minimum element of $(F, \preceq)$. Thus $1 \in \beta(1, x^S) = \overset{-}{\langle}\{v \in x^S \mid v \prec 1\}\overset{+}{\rangle} = \overset{-}{\langle} \varnothing \overset{+}{\rangle} = \{1\}$. Therefore $1 \notin B(x^S)$.

(N1): Assume that there exist $v_1, v_2 \in B(x^S)$ such that $v_1v_2 \neq 1$ and $|v_1v_2| < |v_1|, |v_2|$. We already know that $v_1 \neq v_2$. We can assume that $v_1 \prec v_2$ without loss of generality. Let p denotes the cancellation in the product v_1v_2; there exist a, b and p such that $v_1 = ap, v_2 = p^{-1}b$ and $v_1v_2 = ab$, all reduced. Also, since $v_1, v_2 \in B(x^S) \subset \overset{-}{\langle} x^S \overset{+}{\rangle} \subset x^F \cup (x^{-1})^F$, there exist $s, t \in F, e, f = \pm 1$ such that $v_1 = s^{-1}x^e s, v_2 = t^{-1}x^f t$, all reduced. As $|v_1|$ is odd, $|p| < \frac{1}{2}|v_1|$ or $|p| > \frac{1}{2}|v_1|$ holds, so is $|v_2|$.

(i) Where $|p| < \frac{1}{2}|v_2|$: The cancellation p^{-1} of $v_2 = p^{-1}b = t^{-1}x^f t$ does not contain the center x^f. Therefore there exist $v \in F$ such that $t^{-1} = p^{-1}v^{-1}$, reduced. We see $b = v^{-1}x^f vp$,

$v_2 = p^{-1}v^{-1}x^f vp$, and so $v_1v_2 = ab = av^{-1}x^f vp$, all reduced. Thus $|a| + 2|v| + 1 + |p| = |av^{-1}x^f vp| = |v_1v_2| < |v_2| = |p^{-1}v^{-1}x^f vp| = 2|p| + 2|v| + 1$, and hence $|a| < |p|$. We also see $v_1v_2v_1^{-1} = abp^{-1}a^{-1} = av^{-1}x^f vpp^{-1}a^{-1} = av^{-1}x^f va^{-1}$, reduced. Therefore $|v_1v_2v_1^{-1}| = |av^{-1}x^f va^{-1}| = 2|a| + 2|v| + 1 < |v_2|$, so $v_1v_2v_1^{-1} \prec v_2$. Then $v_1, v_1v_2v_1^{-1} \prec v_2$ and $v_2 = v_1^{-1}v_1v_2v_1^{-1}v_1$. That leads $v_2 \in \beta(v_2, x^S)$, contradicting the definition of $B(x^S)$.

(ii) Where $|p| > \frac{1}{2}|v_2|$: The cancellation p^{-1} of $v_2 = p^{-1}b = t^{-1}x^f t$ contains the center x^f. Therefore there exist $v \in F$ such that $p^{-1} = t^{-1}x^f v$, reduced. Then $v_2 = t^{-1}x^f vb$, reduced. Since x^f and v_2 are conjugate, $t^{-1} = b^{-1}v^{-1}$, so $v_2 = b^{-1}v^{-1}x^f vb$, all reduced.

The assumption $|p| < \frac{1}{2}|v_1|$ leads $|v_2| < 2|p| < |v_1|$, contradicting $v_1 \prec v_2$. Hence $|p| > \frac{1}{2}|v_1|$. The cancellation p of $v_1 = ap = s^{-1}x^e s$ contains the center x^e. Therefore there exist $u \in F$ such that $p = u^{-1}x^e s$, reduced. Then $v_1 = au^{-1}x^e s$, reduced. Since x^e and v_1 are conjugate, $s = ua^{-1}$, so $v_1 = au^{-1}x^e ua^{-1}$, all reduced.

Let q denotes the cancellation in the product $a^{-1}b^{-1}$ and $a^{-1} = cq, b^{-1} = q^{-1}d$, all reduced. Suppose $c \neq 1$ and $d \neq 1$, then $v_1v_2 = au^{-1}x^e ua^{-1}b^{-1}v^{-1}x^f vb = au^{-1}x^e ucdv^{-1}x^f vb$, reduced, and so $|v_1v_2| = |a| + 2|u| + 1 + |c| + |d| + 2|v| + 1 + |b| = 2|p| + |c| + |d| > 2|p| > |v_1|, |v_2|$. This is a contradiction. Therefore $c = 1$ or $d = 1$. Since $v_1 \prec v_2$, we see $|c| + |q| + |p| = |a| + |p| = |v_1| \leq |v_2| = |p| + |b| = |p| + |q| + |d|$. Thus $c = 1$.

If $|v_1| = |v_2|$, then $|a| = |b|$, $0 = |c| = |d|$, $d = 1$, $a^{-1} = q = b$, and so $v_1v_2 = ab = 1$, a contradiction. Therefore $|v_1| < |v_2|$.

Note that $|v| < |u|$ since $|a| < |b|$ and $|p| = |a|+2|u|+1 = |b|+2|v|+1$. As $|v_1| < |v_2|$ and both are odd, $|x^e s| = \frac{|v_1|+1}{2} \leq \frac{|v_2|-1}{2} = |t| < \frac{|v_2|}{2} < |p|$. Thus t, the last $\frac{|v_2|-1}{2}$ letters of the reduced form of $p = u^{-1}x^e s = v^{-1}x^{-f}t$ contains $x^e s$, the last $\frac{|v_1|+1}{2}$ letters of it. There exists w such that $t = wx^e s$, reduced. Then $u^{-1}x^e s = p = v^{-1}x^{-f}t = v^{-1}x^{-f}wx^e s$, all reduced, and hence $u^{-1} = v^{-1}x^{-f}w$, reduced. As a result we get $p = u^{-1}x^e s = u^{-1}x^e ua^{-1} = v^{-1}x^{-f}wx^e w^{-1}x^f va^{-1}$, reduced. Therefore

$$\begin{aligned}
v_2 &= p^{-1}b && (10)\\
&= av^{-1}x^{-f}wx^{-e}w^{-1}x^f vb && (11)\\
&= av^{-1}x^{-f}wx^{-e}w^{-1}x^f t && (12)\\
&= av^{-1}x^{-f}wx^{-e}w^{-1}x^f wx^e s && (13)\\
&= av^{-1}x^{-f}wx^{-e}w^{-1}x^f wx^e ua^{-1} && (14)\\
&= av^{-1}x^{-f}wx^{-e}w^{-1}x^f wx^e w^{-1}x^f va^{-1}, \text{reduced.} && (15)
\end{aligned}$$

As $v_1 = ap = av^{-1}x^{-f}wx^e w^{-1}x^f va^{-1}$, reduced,

$$\begin{aligned}
v_1v_2v_1^{-1} &= av^{-1}x^{-f}wx^e w^{-1}x^f va^{-1}av^{-1}x^{-f}wx^{-e}w^{-1}x^f wx^e w^{-1}x^f va^{-1}v_1^{-1} && (16)\\
&= av^{-1}wx^e w^{-1}x^f va^{-1}v_1^{-1} && (17)\\
&= av^{-1}wx^e w^{-1}x^f va^{-1}av^{-1}x^{-f}wx^{-e}w^{-1}x^f va^{-1} && (18)\\
&= av^{-1}x^f va^{-1}, \text{reduced.} && (19)
\end{aligned}$$

Therefore $|v_1v_2v_1^{-1}| = 2|a| + 2|v| + 1 < 2|a| + 2|v| + 4|w| + 5 = |v_2|$, and so $v_1v_2v_1^{-1} \prec v_2$. Thus $v_1 \prec v_2$, $v_1v_2v_1^{-1} \prec v_2$, and $v_2 = v_1^{-1}v_1v_2v_1^{-1}v_1$ holds. That leads $v_2 \in \beta(v_2, x^S)$, contradicting the definition of $B(x^S)$.

(N2): Assume that there exist $v_1, v_2, v_3 \in B(x^S)$ such that $v_1v_2 \neq 1$, $v_2v_3 \neq 1$ and $|v_1v_2v_3| \leq |v_1| - |v_2| + |v_3|$. As we have checked $B(x^S)$ satisfies (N1), no more than half of the reduced word v_2 can be canceled. Therefore we can write each cancellation as p, q so that $v_1 = ap^{-1}$, $v_2 = pbq^{-1}$ (if $b = 1$, pq^{-1}), $v_3 = qc$ and $v_1v_2v_3 = abc$ (if $b = 1$, ac) , all reduced.

(i) Where $b \neq 1$: In the same way as the proof of $A(U)$, we can reach a contradiction.

(ii) Where $b = 1$: In the proof of $A(U)$ we showed that $|p| = |q|$, and hence $|v_2|$ is even. Meanwhile, all elements of $B(x^S)$ are conjugates of x or x^{-1}. Thus $|v_2|$ is odd, which is a contradiction. □

Proposition 1.3. *Let G be a bi-ordered group. Then $H \trianglelefteq^+ G \implies H \trianglelefteq G$ is false.*

Proof. Let F be the free group generated by $\{a, b\}$. We bi-order F in the way of [4, Definition 3]. To begin with, we order $\mathbb{Z}[[X, Y]]$, the ring of formal power series in the non-commuting variables X, Y. For $f, g \in \mathbb{Z}[[X, Y]]$, we arrange each terms in the lexicographical order where $X < Y$:

$$f = a_0 + a_1X + a_2Y + a_3XX + a_4XY + a_5YX + a_6YY + a_7XXX \cdots \quad (20)$$

$$g = b_0 + b_1X + b_2Y + b_3XX + b_4XY + b_5YX + b_6YY + b_7XXX \cdots . \quad (21)$$

Compare each coefficient from left to right, and declare $f < g$ when $a_i < b_i$ where a_i and b_i are the first pair of coefficients that differ.

Next, we define the Magnus map $\mu : F \to \mathbb{Z}[[X, Y]]$,

$$\begin{cases} a \mapsto 1 + X, & b \mapsto 1 + Y, \\ a^{-1} \mapsto 1 - X + X^2 - X^3 + \cdots , & b^{-1} \mapsto 1 - Y + Y^2 - Y^3 + \cdots . \end{cases} \quad (22)$$

This map is injective [3, Theorem 5.6]. Therefore by defining $u < v$ as $\mu(u) < \mu(v)$ for $u, v \in F$, we obtain a linear order on F. This order is also a bi-order [4, Theorem 4].

The subgroup $\langle b^{F_+} \rangle$ of F is *fee* as shown last time [1]. We will show that $aba^{-1} \notin \langle b^{F_+} \rangle$ and so is not normal. From lemma 1.2, $B(b^{F_+})$ freely generates $\langle b^{F_+} \rangle$.

Firstly, we observe the images of b^F, $(b^F)^{-1}$ and $\langle b^F \rangle$ under μ. Let $s \in F$ and $\mu(s) = 1 + a_1X + a_2Y + a_3X^2 + a_4XY + a_5YX + a_6Y^2 + O(3)$. Then

$$\mu(s^{-1}) = 1 - a_1X - a_2Y + (a_1^2 - a_3)X^2 + (a_1a_2 - a_4)XY + (a_1a_2 - a_5)YX + (a_2^2 - a_6)Y^2 + O(3), \quad (23)$$

and hence

$$\mu(s^{-1}bs) = 1 + Y - a_1XY + a_1YX + O(3), \mu(s^{-1}b^{-1}s) = 1 - Y + a_1XY - a_1YX + O(3). \quad (24)$$

Further, assume that $u, v \in \langle b^F \rangle$, $\mu(u) = 1 + b_0Y - b_1XY + b_1YX + b_2Y^2 + O(3)$ and $\mu(v) = 1 + c_0Y - c_1XY + c_1YX + c_2Y^2 + O(3)$. Then

$$\mu(uv) = 1 + (b_0 + c_0)Y - (b_1 + c_1)XY + (b_1 + c_1)YX + (b_0c_0 + b_2 + c_2)Y^2 + O(3). \quad (25)$$

Thus, for all $u \in \langle b^F \rangle$, there exist c_0, c_1 and $c_2 \in \mathbb{Z}$ such that $\mu(u) = 1 + c_0Y - c_1XY + c_1YX + c_2Y^2 + O(3)$.

Secondly, we will indicate $aba^{-1} \notin B(b^{F_+})$. Suppose $aba^{-1} \in B(b^{F_+})$. Since $B(b^{F_+}) \subset \overset{-}{\langle} b^{F_+} \overset{+}{\rangle}$, there exist $n \geq 1, s_1, s_2, \ldots s_n > 1, e_1, e_2, \ldots e_n = \pm 1$ such that

$$aba^{-1} = s_n^{-1}b^{-e_n}s_n \cdots s_2^{-1}b^{-e_2}s_2s_1^{-1}b^{e_1}s_1s_2^{-1}b^{e_2}s_2 \cdots s_n^{-1}b^{e_n}s_n. \quad (26)$$

Then $e_1 = 1$ because b^{e_1} is the central letter of aba^{-1}. (This can also be checked by observing the images of both sides of the equation under μ.) Let $\mu(s_1) = 1 + a_1X + a_2Y + a_3X^2 + a_4XY + a_5YX + a_6Y^2 + O(3)$, then $\mu(s_1^{-1}b^{e_1}s_1) = \mu(s_1^{-1}bs_1) = 1 + Y - a_1XY + a_1YX + O(3)$. Moreover, since $s_2^{-1}b^{e_2}s_2 \cdots s_n^{-1}b^{e_n}s_n \in \langle b^{F_+}\rangle \subset \langle b^F\rangle$, we can write $\mu(s_2^{-1}b^{e_2}s_2 \cdots s_n^{-1}b^{e_n}s_n) = 1 + c_0Y - c_1XY + c_1YX + c_2Y^2 + O(3)$. Then $\mu(s_n^{-1}b^{-e_n}s_n \cdots s_2^{-1}b^{-e_2}s_2) = 1 - c_0Y + c_1XY - c_1YX + (c_0^2 - c_2)Y^2 + O(3)$, and hence

$$\mu(aba^{-1}) = \mu(s_n^{-1}b^{-e_n}s_n \cdots s_2^{-1}b^{-e_2}s_2)\mu(s_1^{-1}bs_1)\mu(s_2^{-1}b^{e_2}s_2 \cdots s_n^{-1}b^{e_n}s_n) \tag{27}$$

$$= 1 + Y - a_1XY + a_1YX + O(3). \tag{28}$$

Meanwhile, since

$$\mu(aba^{-1}) = \mu(a)\mu(b)\mu(a^{-1}) \tag{29}$$

$$= (1 + X)(1 + Y)(1 - X + X^2 + O(3)) \tag{30}$$

$$= 1 + Y + XY - YX + O(3), \tag{31}$$

by comparing the coefficients of (28) and (31), we see $a_1 = -1$. Therefore $\mu(s_1) = 1 + a_1X + a_2Y + O(2) = 1 - X + a_2Y + O(2) < 1 = \mu(1)$, and so $s_1 < 1$. This contradicts $s_1 > 1$.

Thirdly, we will indicate $aba^{-1} \notin \langle b^{F_+}\rangle = \langle B(b^{F_+})\rangle$. Suppose $aba^{-1} \in \langle b^{F_+}\rangle$, that is, there exist $t \in \mathbb{N}, u_1, \ldots, u_t \in B(b^{F_+})$ such that $aba^{-1} = u_1 \cdots u_t$ and all $u_iu_{i+1} \neq 1$. Then, as shown in [5, Proposition 2.3], $t \leq 3$, $|u_1|, |u_t| \leq 4$. Since $|u_1|$ and $|u_t|$ are odd, $|u_1|, |u_t| \leq 3$. If $t = 3$, $|u_2|$ is also at most 3, because $B(b^{F_+})$ is N-reduced, so at most 1 letter is canceled from each of u_1 and u_3, and hence at most 2 letters are canceled from u_2.

We see that $B(b^{F_+}) \subset \overset{-}{\langle} b^{F_+} \overset{+}{\rangle} \subset (b^{\pm 1})^F$. All the elements of $(b^{\pm 1})^F$ whose length is at most 3 are $1, b, a^{-1}ba, aba^{-1}$ up to inversion. We already checked that $aba^{-1} \notin B(b^{F_+})$. All the products $u_1 \cdots u_t$ where $1 \leq t \leq 3$, all $u_i = b^{\pm 1}$ or $(a^{-1}ba)^{\pm 1}$, and $u_iu_{i+1} \neq 1$ are the followings:

		b^3			b^{-3}
	b^2	$b^2a^{-1}ba$		b^{-2}	$b^{-2}a^{-1}ba$
		$b^2a^{-1}b^{-1}a$			$b^{-2}a^{-1}b^{-1}a$
		$ba^{-1}bab$			$b^{-1}a^{-1}bab$
b	$ba^{-1}ba$	$ba^{-1}bab^{-1}$	b^{-1}	$b^{-1}a^{-1}ba$	$b^{-1}a^{-1}bab^{-1}$
		$ba^{-1}b^2a$			$b^{-1}a^{-1}b^2a$
		$ba^{-1}b^{-1}ab$			$b^{-1}a^{-1}b^{-1}ab$
	$ba^{-1}b^{-1}a$	$ba^{-1}b^{-1}ab^{-1}$		$b^{-1}a^{-1}b^{-1}a$	$b^{-1}a^{-1}b^{-1}ab^{-1}$
		$ba^{-1}b^{-2}a$			$b^{-1}a^{-1}b^{-2}a$
		$a^{-1}bab^2$			$a^{-1}b^{-1}ab^2$
	$a^{-1}bab$	$a^{-1}baba^{-1}ba$		$a^{-1}b^{-1}ab$	$a^{-1}b^{-1}aba^{-1}ba$
		$a^{-1}baba^{-1}b^{-1}a$			$a^{-1}b^{-1}aba^{-1}b^{-1}a$
		$a^{-1}bab^{-2}$			$a^{-1}b^{-1}ab^{-2}$
$a^{-1}ba$	$a^{-1}bab^{-1}$	$a^{-1}bab^{-1}a^{-1}ba$	$a^{-1}b^{-1}a$	$a^{-1}b^{-1}ab^{-1}$	$a^{-1}b^{-1}ab^{-1}a^{-1}ba$
		$a^{-1}bab^{-1}a^{-1}b^{-1}a$			$a^{-1}b^{-1}ab^{-1}a^{-1}b^{-1}a$
		$a^{-1}b^2ab$			$a^{-1}b^{-2}ab$
	$a^{-1}b^2a$	$a^{-1}b^2ab^{-1}$		$a^{-1}b^{-2}a$	$a^{-1}b^{-2}ab^{-1}$
		$a^{-1}b^3a$			$a^{-1}b^{-3}a$

(32)

There is not aba^{-1} on this list. Thus $aba^{-1} \notin \langle b^{F_+}\rangle$, and hence $\langle b^{F_+}\rangle$ is not normal.

□

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