Discriminating whether a function of a real variable is an iterated function

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Abstract

Suppose a be a sequence and there exists a map f such that $a_n = f^n(a_0)$ for all $n \in \mathbb{N}$. Then for any $l \in \mathbb{N}$, $a_m = a_n$ implies $a_{l+m} = f^l(a_m) = f^l(a_n) = a_{l+n}$. We will generalize the idea to a its domain is an linearly ordered group.

1 Main Text

In this article we discuss the case a map a defined on a linearly ordered group G or a subset of G which closed upwards satisfies

$$a(g) = a(h) \implies a(fg) = a(fh) \text{ and } a(gf) = a(hf) \quad (f \in G_+), \tag{1}$$

where G_+ is the set of positives of G. If the G is commutative, we immediately get

$$a(np + g + h) = a(g + h) \quad (n \in \mathbb{N}, g \in G_{\geq 0}), \tag{2}$$

where $G_{\geq 0}$ denotes $\{0\} \cup G_+$.

We use "LO", "RO", and "BO" as abbreviations for "left-ordered", "right-ordered" and "bi-ordered".

Definition 1.1. If a is a map, we let $\sim_a$ denotes equivalence kernel of a: $x \sim_a y :\iff a(x) = a(y)$. Also we use $a \simeq b$ to denote the condition that a and b, maps defined on same domain, are satisfying $a(x) = a(y) \iff b(x) = b(y)$.

Definition 1.2. For a map a whose domain is a left-ordered group G, we define the *periods* of a, denoted $P(a)$, as

$$P(a) := \{g^{-1}h \in G \mid a(g) = a(h), g < h\}$$
$$= \{g^{-1}h \in G \mid a(g) = a(gg^{-1}h), g < h\} = \{p \in G_+ \mid \exists h \in G, a(h) = a(hp)\}$$

and the *heights* of a, denoted $H(a)$, as

$$H(a) := \{h \in G \mid \exists p \in G_+, a(h) = a(hp)\}.$$

From this definition,

$$
\begin{aligned}
&\{g^{-1}h \in G \mid g \sim_a h\}\\
&= \{g^{-1}h \in G \mid g \sim_a h, g < h\} \sqcup \{1\} \sqcup \{g^{-1}h \in G \mid g \sim_a h, g > h\}\\
&= \{g^{-1}h \in G \mid g \sim_a h, g < h\} \sqcup \{1\} \sqcup \{(h^{-1}g)^{-1} \in G \mid g \sim_a h, g > h\}\\
&= \{g^{-1}h \in G \mid g \sim_a h, g < h\} \sqcup \{1\} \sqcup \{g^{-1}h \in G \mid g \sim_a h, g < h\}^{-1}\\
&= P(a) \sqcup \{1\} \sqcup P(a)^{-1}.
\end{aligned}
$$

Lemma 1.3. *Let G be BO, a map a that its domain is G holds (1), and $F := \{g^{-1}h \in G \mid a(g) = a(h)\} = P(a) \sqcup \{1\} \sqcup P(a)^{-1}$. Then F is a subgroup of G. Moreover*

$$g^{-1}Fg \subset F \text{ for all } g \in G_+. \tag{3}$$

Proof. For all $p, q \in P(a)$, there are $g, h \in G$ s.t. $g \sim_a gp$, $h \sim_a hq$. We can assume $g \leq h$ without loss of generality. Then, since G is RO, $hg^{-1} \in G_{\geq 1}$. By using (1) to $g \sim_a gp$, $h = hg^{-1}g \sim_a hg^{-1}gp = hp$. We get that for all $p, q \in P(a)$, there exists $h \in G$ s.t.

$$h \sim_a hp \sim_a hq. \tag{4}$$

Now we check that the F is closed under products. For all $p, q \in F$, if $p = 1$ or $q = 1$, trivially $pq \in F$.
(i) Where $p, q \in P(a)$: From (4), there is a $g \in G$ s.t. $g \sim_a gp \sim_a gq$. Since a holds (1), by multiplying both sides of $g \sim_a gp$ by q, we see $gq \sim_a gpq$. Thus $g \sim_a gpq$. That is $pq \in P(a) \subset F$.
(ii) Where $p, q \in P(a)^{-1}$: $p^{-1}, q^{-1} \in P(a)$. We have already checked that $q^{-1}p^{-1} \in P(a)$ as above. Therefore $pq = (q^{-1}p^{-1})^{-1} \in P(a)^{-1} \subset F$.
(iii) Where $p \in P(a)^{-1}$, $q \in P(a)$:
(iii-i) Where $pq = 1$: $pq \in \{1\} \subset P(a)$.
(iii-ii) Where $pq > 1$: From (4), there is a $g \in G$ s.t. $g \sim_a gp^{-1} \sim_a gq$. Multiplying both sides of $g \sim_a gp^{-1}$ on the right by pq, we have $gpq \sim_a gp^{-1}pq = gq$.
(iii-iii) Where $pq < 1$: Multiplying both sides of $g \sim_a gq$ on the right by $q^{-1}p^{-1} = (pq)^{-1} > 1$, we have $gq^{-1}p^{-1} \sim_a gqq^{-1}p^{-1} = gp^{-1}$. Thus $g(pq)^{-1} \sim_a gp^{-1} \sim_a g$. That is $pq \in P(a)^{-1} \subset F$.
(iv) Where $p \in P(a)$, $q \in P(a)^{-1}$:
(iv-i) Where $pq = 1$: $pq \in \{1\} \subset F$.
(iv-ii) Where $pq > 1$: Because G is BO if and only if $g^{-1}G_+g \subset G_+$ for all $g \in G$, $gpqg^{-1} > 1$. From (4), there is a $g \in G$ s.t. $g \sim_a gp \sim_a gq^{-1}$. Multiplying both sides of $g \sim_a gq^{-1}$ on the left by $gpqg^{-1}$, we have $gpq = gpqg^{-1}g \sim_a gpqg^{-1}gq^{-1} = gp$.
(iv-iii) Where $pq < 1$: Since G is BO, $g(pq)^{-1}g^{-1} = gq^{-1}p^{-1}g^{-1} > 1$. Multiplying both sides of $g \sim_a gp$ on the left by $gq^{-1}p^{-1}g^{-1}$, we have $gq^{-1}p^{-1} = gq^{-1}p^{-1}g^{-1}g \sim_a gq^{-1}p^{-1}g^{-1}gp = gq^{-1}$. Thus $g(pq)^{-1} \sim_a gp \sim_a g$. That is $pq \in P(a)^{-1} \subset F$.

Now we check that $g^{-1}Fg \subset F$ for all $g \in G_+$. For all $p \in F$ and $g \in G_+$, if $p \in P(a)$, there is a $h \in G$ s.t. $h \sim_a hp$. Using (1), we see $hg \sim_a hpg$. From the definition of $P(a)$, $g^{-1}pg = (hg)^{-1}(hpg) \in P(a) \subset F$. If $p = 1$, $1 = g^{-1}1g \in \{1\} \subset F$. If $p \in P(a)^{-1}$, $p^{-1} \in P(a)$ and therefore $g^{-1}p^{-1}g = (hg)^{-1}(hp^{-1}g) \in P(a)$ as above. Hence $(g^{-1}p^{-1}g)^{-1} = g^{-1}pg \in P(a)^{-1} \subset F$. □

Proposition 1.4. *Let G be a linearly ordered group and $S \subset G$. Suppose F_0 is the smallest subgroup of G such that contains S and holds (3). Then*

$$F_0 = \bigcap \mathcal{F} = \langle S^{G_+} \rangle,$$

where $\mathcal{F} = \{F \mid S \subset F \leq G, F \text{ holds (3)}\}$, $S^{G_+} = \{g^{-1}sg \mid s \in S, g \in G_+\}$.

Proof. We see $\mathcal{F} \neq \varnothing$ as G holds (3). Clearly $\bigcap \mathcal{F}$ is a subgroup of G. For all $f \in \bigcap \mathcal{F}$ and $g \in G_+$, considering $f \in F$ for any $F \in \mathcal{F}$, $g^{-1}fg \in F$ for any $F \in \mathcal{F}$. Thus $g^{-1}fg \in \bigcap \mathcal{F}$.

Given S, $\langle S^{G_+} \rangle$ is the smallest subgroup containing S^{G_+}. For all $f \in \langle S^{G_+} \rangle$ and $g \in G_+$, the f can be written as

$$f = g_1^{-1} s_1^{\epsilon_1} g_1 \dots g_n^{-1} s_n^{\epsilon_n} g_n$$

with $n \in \mathbb{N}$, $s_1, \dots, s_n \in S$, $\epsilon_1, \dots \epsilon_n \in \{1, -1\}$ and $g_1, \dots, g_n \in G_+$. Then

$$\begin{aligned} g^{-1}fg &= g^{-1}g_1^{-1} s_1^{\epsilon_1} g_1 g_2^{-1} s_2^{\epsilon_2} g_2 \dots g_n^{-1} s_n^{\epsilon_n} g_n g \\ &= g^{-1}g_1^{-1} s_1^{\epsilon_1} g_1 1 g_2^{-1} s_2^{\epsilon_2} g_2 1 \dots 1 g_n^{-1} s_n^{\epsilon_n} g_n g \\ &= g^{-1}g_1^{-1} s_1^{\epsilon_1} g_1 g g^{-1} g_2^{-1} s_2^{\epsilon_2} g_2 g \dots g^{-1} g_n^{-1} s_n^{\epsilon_n} g_n g \\ &= (g_1 g)^{-1} s_1^{\epsilon_1} (g_1 g)(g_2 g)^{-1} s_2^{\epsilon_2} (g_2 g) \dots (g_n g)^{-1} s_n^{\epsilon_n} (g_n g). \end{aligned}$$

As $g_i g \in G_+$ for all $1 \leq i \leq n$, $g^{-1}fg \in \langle S^{G_+} \rangle$.

Since $\langle S^{G_+} \rangle$ is a subgroup satisfying (3), we have $\bigcap \mathcal{F} \subset \langle S^{G_+} \rangle$. By definition of $\bigcap \mathcal{F}$, $S \subset \bigcap \mathcal{F}$. Considering $\bigcap \mathcal{F}$ is a subgroup which holds (3), $\langle S^{G_+} \rangle \subset \bigcap \mathcal{F}$. □

Lemma 1.5. *Let G be a bi-ordered group, a map a on G holds (1), $h \in H(a)$. Then $k \in H(a)$ for all $k > h$.*

Proof. As $h \in H(a)$, there is a $p \in G_+$ s.t. $a(h) = a(hp)$. In RO group, $k > h$ implies $kh^{-1} > 0$. Multiplying both sides of $h \sim_a hp$ on the left by kh^{-1}, we have $k \sim_a p + k$. □

Corollary 1.6. *Let G be a bi-ordered group, a map a on G holds (1). Then the following three are hold:*

- *(i)* $H(a) = \varnothing \iff$ *a is injective.*
- *(ii)* $H(a) = G \iff$ *a is not injective and $H(a)$ is unbounded below.*
- *(iii)* $(G \backslash H(a), H(a))$ *is a Dedekind cut of G* $\iff$ *a is not injective and $H(a)$ is bounded below.*

So $H(a)$ is $\varnothing$, G, or an upper set of a cut.

Lemma 1.7. *Let G be an additive subgroup of $\mathbb{R}$, a map a on G holds (1). Then*

$$\{(p,h) \in G_+ \times G \mid a(h) = a(p+h)\} = P(a) \times H(a).$$

Proof. For all $(p,h) \in P(a) \times H(a)$, there exist $k \in G$ and $q \in G_+$ s.t. $a(k) = a(p+k)$ and $a(h) = a(q+h)$.
(i) Where $h \geq k$: We have already checked that $h \sim_a p+h$ holds in the process of deriving (4).
(ii) Where $h \leq k$: Because G is an Archimedean group, there exists $n \in \mathbb{N}$ s.t. $nq \geq k - h$. Therefore

$$\begin{aligned}
&a(p+h) \\
&= a(nq+p+h) && \text{by (2) with } g = p \in G_{\geq 0} \\
&= a(nq+p+h-k+k) \\
&= a(p+(nq+h-k)+k) \\
&= a((nq+h-k)+k) && \text{by (2) with } n = 1 \text{ and } g = nq+h-k \in G_{\geq 0} \\
&= a(nq+h) \\
&= a(h) && \text{by (2).}
\end{aligned}$$

□

Lemma 1.8. *Let $\pm P_0(a)$ denotes $-P(a) \sqcup \{0\} \sqcup P(a)$. Suppose G is a subgroup of $(\mathbb{R}, +)$, a is a map defined on G satisfying (1), and π is the canonical surjection $G \to G/\pm P_0(a)$. Define a map b on G as*

$$b(g) := \begin{cases} (g, 1) & \text{if } g \notin H(a) \\ (\pi(g), 2) & \text{if } g \in H(a). \end{cases}$$

Then $a \simeq b$.

Proof. By Lemma 1.3, $\pm P_0(a)$ is a subgroup of G. Thus $G/\pm P_0(a)$ is defined as a quotient group and so there exists a canonical surjection π. To derive $a(g) = a(h) \iff b(g) = b(h)$, we only need to check the case where $g < h$.

Suppose $a(g) = a(h)$. Since $g < h$, $-g+h \in P(a)$, and therefore $\pi(g) = \pi(h)$. As $a(g) = a(h) = a(g-g+h)$, $g \in H(a)$. By Lemma 1.5, $h \in H(a)$, so $b(g) = (\pi(g), 2) = (\pi(h), 2) = b(h)$.

Suppose $b(g) = b(h)$. If $g \notin H(a)$, $b(g) = b(h) = (g, 1) \in G$. By definition of b, $h \notin H(a)$ and hence $(h, 1) = b(h) = b(g) = (g, 1)$, contradicting $g < h$. Thus $g \in H(a)$, so is h. As $b(g) = b(h)$, we get $\pi(g) = \pi(h)$: $-g+h \in P(a)$. By Lemma 1.7, since $g \in H(a)$ and $-g+h \in P(a)$, $a(g) = a(g-g+h) = a(h)$. □

Lemma 1.9. *Suppose G is a subgroup of $(\mathbb{R}, +)$, H is a subgroup of H, and (α, β) is a Dedekind cut of G. The canonical surjection $\pi : G \to G/H$ and the map b defined as*

$$b(g) := \begin{cases} (g, 1) & \text{if } g \in \alpha \\ (\pi(g), 2) & \text{if } g \in \beta \end{cases} \tag{5}$$

are satisfying (1).

Proof. It is clear that π is. To prove $b(g) = b(h) \implies b(f+g) = b(f+h) \quad (f \in G_+)$, we only need to check the case where $g < h$. Considering the definition of b, $g, h \in \beta$. Thus $(\pi(g), 2) = b(g) = b(h) = (\pi(h), 2)$, so $\pi(g) = \pi(h)$. Because β is the upper set of the cut, for any $f \in G_+$, $f+g, f+g \in \beta$. Hence $b(f+g) = (\pi(f+g), 2) = (\pi(f+h), 2) = b(f+h)$. □

Proposition 1.10. *Suppose a is a map defined on a subgroup G of $(\mathbb{R}, +)$. By Lemma 1.5, 1.8 and 1.9, a satisfies (1) if and only if there are a subgroup H of G and a cut (α, β) of G such that $a \simeq \pi$ or $a \simeq b$ holds, where π is the canonical surjection $G \to G/H$ and b is defined as (5).*

Proposition 1.11. *Suppose G is a subgroup of $(\mathbb{R}, +)$, (α, β) is a cut of G, a is a map defined on β. Then a satisfies (1) if and only if the map A defined as*

$$A(g) := \begin{cases} (g, 1) & \text{if } g \in \alpha \\ (a(g), 2) & \text{if } g \in \beta \end{cases}$$

satisfies (1).

Proof. Suppose a satisfies (1). To prove $A(g) = A(h) \implies A(f+g) = A(f+h) \quad (f \in G_+)$, we can assume $g \leq h$. If $g \in \alpha$, as $A(g) = A(h) = (g, 1)$, we see $g = h$. If $g \in \beta$, $h \in \beta$. Thus $(a(g), 2) = A(g) = A(h) = (a(h), 2)$. Since $f+g, f+h \in \beta$, for all $f \in G_+$, $A(f+g) = (a(f+g), 2) = (a(f+h), 2) = A(f+h)$. □

Exapmle 1.12. Define an operator and an ordering on $\mathbb{R} \times \mathbb{Z}$ as $(x, m) + (y, n) := (x+m, y+n)$ and $(x, m) \leq (y, n) :\iff m < n$ or $(m = n$ and $x \leq y)$. The map $a : \mathbb{R} \times \mathbb{Z} \to \bigcup_{n \in \mathbb{Z}} ((\mathbb{R}/2^{-n}\mathbb{Z}) \times \{n\})$ defined as $a(x, m) := (x + 2^{-m}\mathbb{Z}, m)$ satisfies (1).

Proof. Suppose $a(x+m) = a(y+n)$. Then trivially $m = n$. As $x + 2^{-m}\mathbb{Z} = y + 2^{-m}\mathbb{Z}$, there is a $u \in \mathbb{Z}$ s.t. $x - y = 2^{-m}u$.

For all $(w, l) \in (\mathbb{R} \times \mathbb{Z})_+$, we know $l \geq 0$ and hence $2^l \in \mathbb{Z}$. Therefore $(w+x) - (w+y) = x - y = 2^{-m}u = 2^{-(l+m)}2^l u$, so $a(w+x, l+m) = a(w+y, l+n)$. □

On Archimedean groups, that is subgroups of $(\mathbb{R}, +)$, we can classify maps satisfying (1) by $P(a)$ and $H(a)$. On this example, since the group $\mathbb{R} \times \mathbb{Z}$ is not Archimedean, we can not use the method.

Postscript

On Lemma 1.3, we showed that for every a satisfying (1), $P(a)$ is a subgroup, but also holds (3). We see all normal subgroups fulfill (3). Is $P(a)$ normal for any such a? I don't think so, and to show a counterexample, I defined

$$\mathrm{fcl}(y) := \text{the smallest subgroup s.t. contains } y \text{ and holds (3)} = \langle \{u^{-1}yu \mid u \in F_+\} \rangle$$

using the bi-ordered free group $F = \langle x, y \rangle$ in [1]. I tried to prove that $\mathrm{fcl}(y)$ does not contain xyx^{-1} (and hence $\mathrm{fcl}(y)$ is not normal), but finally I couldn't find the proof, and I finished this article.

17/10/2024: Corrected (fh) and hp to $a(fh)$ and $a(hp)$ in (1) and Definition 1.2. Corrected $g^{-1}h \in G_+$ to $g^{-1}h \in G$ in Definition 1.2 and the following. Corrected $P(a) \cup \{1\} \cup P(a)^{-1}$ and $P(a) \cup \{0\} \cup P(a)^{-1}$ to $P(a) \sqcup \{1\} \sqcup P(a)^{-1}$ and $P(a) \sqcup \{0\} \sqcup P(a)^{-1}$ in Lemma 1.3 and 1.8. Corrected $\{1\} \subset P(a)$ to $\{1\} \subset F$ in 1.3 (iv-i). Corrected $g^{-1}Pg \subset P$ to $g^{-1}G_+g \subset G_+$ in Lemma 1.3 (iv-ii). Added "We use "LO", "RO", and "BO" as abbreviations for "left-ordered", "right-ordered" and "bi-ordered"." in p. 1.

References

[1] Clay, A. & Rolfsen, D. (2015), *Ordered Groups and Topology,* `https://arxiv.org/abs/1511.05088`