$^2\mathbb{R}$. FRACVECTORS

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1. Introduction

This document introduces the novel mathematical structure of *fracvectors*, which generalize vectors to fractional dimensions. Fracvectors extend the Clifford Algebra by allowing the use of blades with fractional grade (dimension).

The main motivation for this work is to apply this new framework to the study of tetration, a process consisting in the repeated exponentiation of a number. By representing numbers as vectors in a fractional-dimensional space, the aim is to gain a deeper understanding of the properties of tetration and its applications.

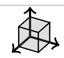
Operation	**Set**	**Dimension**	**Generalization**	**Representation**
-	$\{1\}$	0		•
addition	$\mathbb{Z} \equiv \mathbb{R}^0$	0	replication	• • • • •
multiplication	$\mathbb{R} \equiv \mathbb{R}^1$	1	interpolation	•—•—•—•→
exponentiation	$\mathbb{R}^{\mathbb{N}}$ (or $\mathbb{R}^{\mathbb{Z}}$?)	$\mathbb{N}$ (or $\mathbb{Z}$?)	replication	
tetration	$\mathbb{R}^{\mathbb{R}} \equiv {}^2\mathbb{R}$?	$\mathbb{R}$?	interpolation?	

Table 1. Hyperoperations and the sets they introduce.

It is hypothesized that each hyperoperation necessitates an expansion in the set of numbers it encompasses, which occurs in two distinct phases as shown in Table 1. The initial phase involves the replication of existing number sets into multiple copies, while the subsequent phase interpolates between these established sets.

Addition takes the set containing the number 1, which is zero-dimensional, and extends to the integers, were each element is a replica of the original set, resulting in a new set, also characterized by a zero Hausdorff dimension.

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Multiplication interpolates between integer points to form a one-dimensional continuum, $\mathbb{R}$.

Exponentiation requires the introduction of elements squaring to -1, such as imaginary units. This extension broadens to include hypercomplex numbers and the Clifford algebra $\mathcal{C}\ell_{n,0,0}$, $n \in \mathbb{N}$, which are defined over dimensions represented by natural numbers.[1 2]

The geometric product of distinct elements in $\mathcal{C}\ell_{n,0,0}$, introduces new elements, such as bivectors and trivectors. These are not mere replicas of the real line but combinations of elements from different axis isomorphic to the real line, known as blades of different grades in a multivector. That is why the visual representation in Table 1 (page 1) includes three axes (illustrating these replicas) and also a cube, which represents new blades formed as the combinations.

Building on this progression, tetration is proposed to interpolate further between $\mathcal{C}\ell_{n,0,0}$ structures, thereby extending the numerical framework to potentially include elements defined over real dimensions, suggested to be denoted as $\mathbb{R}^{\mathbb{R}} \equiv {}^{2}\mathbb{R}$. [3]

This perspective on fractional dimensions through fracvectors offers a new lens through which to explore the meaning of tetration to fractional exponents.

2. Fracvectors

Definition 1 (Fracvector)**.** *A* fracvector *is a set of infinite points, whose coordinates are expressed in a numeral system in base b, restricted to using only b^D distinct digits, where D is the dimension of the fracvector. Each digit in this string is a vector pointing to a specific spatial position.*

The fracvector is a fractal, that when scaled by b, it produces b^D copies of the original fractal. Each copy is positioned according to the vector represented by the corresponding digit in the fracvector, pointing from the origin $(0, 0)$ of the basis fracversor to the origin of the copy. If b is a complex number, each copy may be rotated respect to the original.

Definition 2 (Fracversor)**.** *The maximum number of points representable inside the unit circle is a fracversor. It is the "unit" fracvector. His digits indicate the points where copies* ***would*** *be located if the fracversor were scaled by the basis b, thus the basis fracversor not necessarily include the points indicated in his digits.*

The dimension of the fracvector, according to the Hausdorff definition, is $D = \frac{log(b^D)}{log(b)}$.

The string representation includes the fractional part, generally represented to the right of the decimal point. As an extension of a Clifford Algebra, the fracvector is a blade of grade $D \in \mathbb{Q}$, oriented, and represents a *quantity* of fractal, without a specific spatial shape.

Figure 1 shows, in red color, the digits used on each base.

Note that the labels in the x axis are enumerating intervals, but the entire image shows only the numeric interval [0,1).

The two first images represent the same, continuous, unit interval $[0, 1)$, in base $b = 10$, and in base $b = 2$. The segments $0.5_{10}, 0.6_{10}, 0.7_{10}, 0.8_{10}, 0.9_{10}$ all together, correspond to the segment 0.1_2 in the second image.

[1]It is generally believed that exponentiation only introduces complex numbers, but actually any unit that squares to -1 satisfies the requirements to assure closure under exponentiation. In $\mathcal{C}\ell_{n,0,0}$ there is an infinite number of imaginary units. This could be generalized to $\mathcal{C}\ell_{n,m,0}$, which explicitly introduces negative squaring units, but it's chosen to focus on $\mathcal{C}\ell_{n,0,0}$ for simplicity.

[2]However, it's worth considering whether mathematicians had an oversight in not considering this generalization to $\mathbb{R}^{\mathbb{Z}}$ instead of $\mathbb{R}^{\mathbb{N}}$, allowing for negative dimensions.

[3]Also, ${}^{2}\mathbb{R}$ possibly generalizes to ${}^{\mathbb{N}}\mathbb{R}$ (or ${}^{\mathbb{Z}}\mathbb{R}$?), same as $\mathbb{C} \equiv \mathbb{R}^2$ generalizes to $\mathbb{R}^{\mathbb{N}}$.

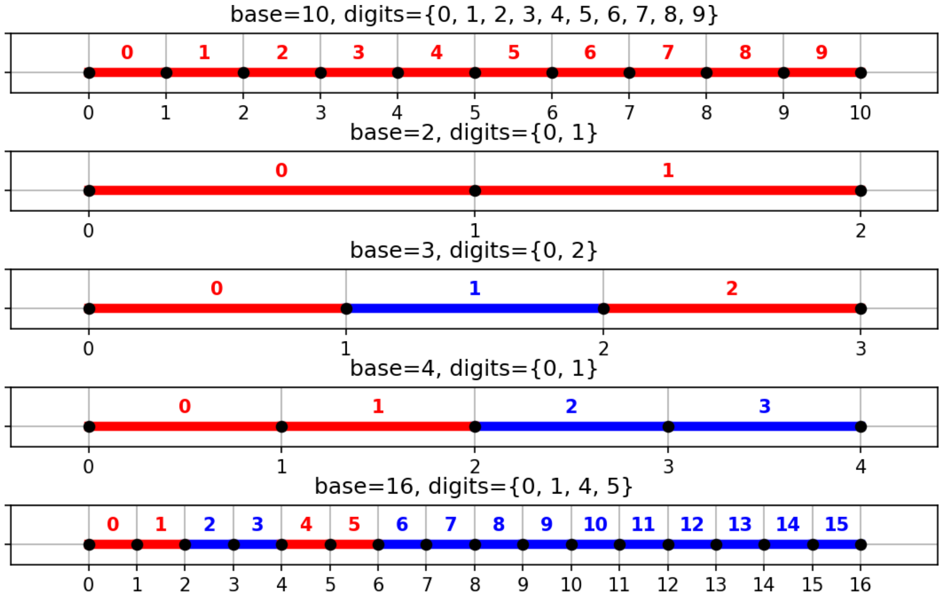

FIGURE 1. Digits used on each base

The next one corresponds to the digits used in the Cantor set. Since the central segment is discarded, it corresponds to banning the use of the digit 1, when representing the unit interval in base 3.

The last two graphics also represent the same fracversor, in base $b = 4$, and in base $b = 16$. They are the same fracvector, because each red segment is subdivided in b subsegments, of length $\frac{1}{b}$, where only the allowed digits are use to address the points. This is because $0.1_4 = 0.4_{16}$ (0.1 in base 4 is the same umber as 0.4 in base 16). The segments $0.4_{16}, 0.5_{16}$ are the segments $0.10_4, 0.11_4$ and the segments $0.6_{16}, 0.7_{16}$ are the non addressable segments $0.12_4, 0.13_4$.

Notation:

> The fracversor is annotated $f(\{\text{List of digits}\}, \text{Base})$. The zero is always present, so it is implicitly taken as present if it is not included in the list of digits.

Examples:

A "right" Sierpinski triangle[4], generated by $f(\{0,1,i\},2)$, when scaled by $b=2$, produces $b^D=3$ copies of the original fracversor. The original fracversor is located at coordinates $(0,0)$, becoming the first copy, and **if it were scaled** by $b=2$, the other two copies would be located at the coordinates indicated by the digits $1=(1,0)$ and $i=(0,1)$. The coordinates of each point can be described by a number written in base 2, using any of the digits $\{0,1,i\}$. Note that in the Figure

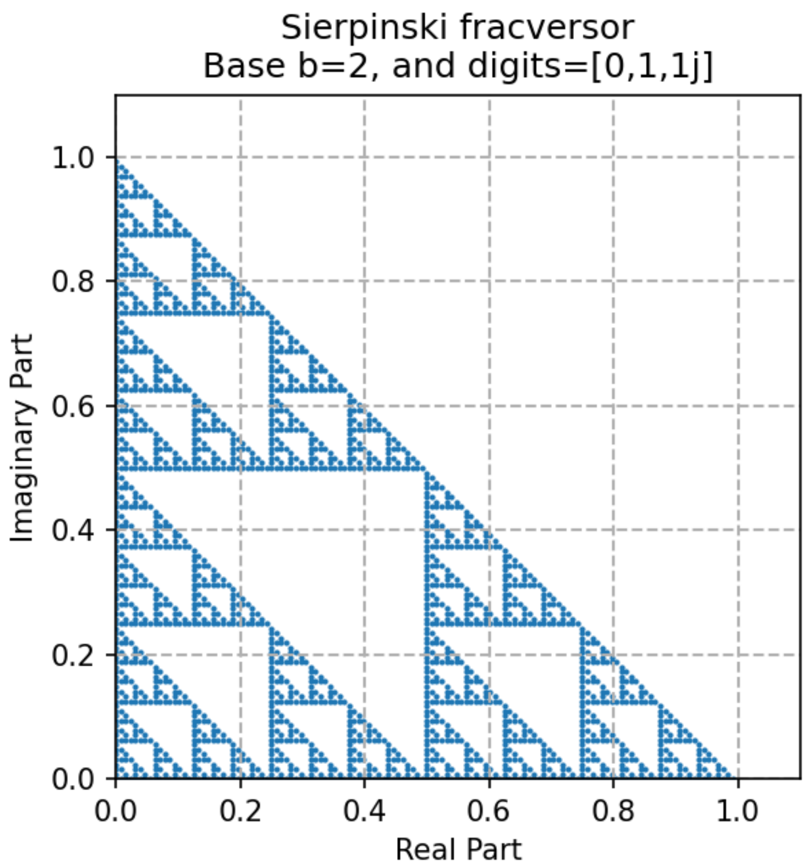

FIGURE 2. Scaling a Sierpinski triangle by $s=2$, multiplies it by $s^{\frac{\log(3)}{\log(2)}}=3$.

2, the imaginary unit has been written $1j$, to avoid the confusion with the number $1i=1\cdot 2+i\cdot 2^0$. Some points like the one at coordinates $(0.5,0.5)$ can be written in binary, in two different ways: $0.1iiiiiiii\ldots_2$, or $0.i111111\ldots_2$.

$$(0.5_{10},0)=0.1_2=0\cdot 2^0+1\cdot 2^{-1}$$

$$(0,0.5_{10})=0.1_2j=0\cdot 2^0+i\cdot 2^{-1}$$

$$(0.5_{10},0,5_{10})=0\cdot 2^0+1\cdot 2^{-1}+i\cdot 2^{-2}+i\cdot 2^{-3}+\cdots=\texttt{0.1iiiiiii}_2\ldots$$

$$(0.5_{10},0,5_{10})=0\cdot 2^0+i\cdot 2^{-1}+1\cdot 2^{-2}+1\cdot 2^{-3}+\cdots=\texttt{0.i11111111}_2\ldots$$

[4]The classic Sierpinski triangle is generally contained in an isosceles triangle, so it is generated by the fracversor $f(\{0,1,\frac{1+i}{\sqrt{2}}\},2)$. Here the word "right" is used in the same way that an orthogonal triangle is called, to refer to the shape of the Sierpinski triangle in Figure 2.

The Cantor set, generated by $f(\{0,2\},3)$, when scaled by $b^D = 3$, generates 2 copies of the original. The original fracversor is located at coordinates $(0,0)$, becoming the first copy, and the other copy is located at coordinates $(2,0)$. The coordinates of each point can be described by a number written in base 3, using any of the digits $\{0,2\}$. The absence of the digit 1 is equivalent to the remotion of the central segment used traditionally to construct the Cantor set.

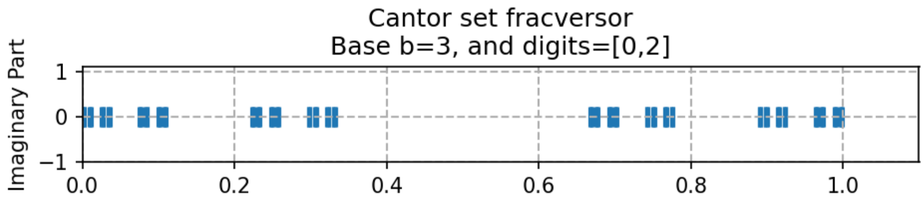

FIGURE 3. Scaling a Cantor set by $s = 3$, multiplies it by $s^{\frac{log(2)}{log(3)}} = 2$.

2.1. **Orthonormal basis.**

Definition 3 (Orthonormal basis)**.** *Given a base b, and a dimension D, an* orthonormal basis *is a set of b^D fracversors of dimension D with b^D digits, only having* 0 *as common digit.*

The equation (1) decomposes a number r as a sum of points from each of the b^D fracversors of the base. The coefficient a_k of each point is in $\{0,1\}$, and the point f_k is in the k^{th} fracversor of the base. The base is orthonormal because this decomposition is unique, meaning that there is only one set of coefficients a_k that can be used to write a given real number r as a linear combination of the points in the base. This is possible because by definition, a fracvector is only parallel to a fracversor only if it has points from that fracversor scaled by b^n, $n \in \mathbb{Z}$; any other scaling number would produce a different fracversor or combination of fracversors.

$$r = \sum_{k=1}^{\frac{1}{D}} a_k b^n f_k \tag{1}$$

Examples:

Dimension	Base b	# digits	Orthonormal basis		
$D = \frac{1}{2}$	4	$b^{\frac{1}{2}} = 2$	$f(\{0,1\},4)$	$f(\{0,2\},4)$	
	4	$b^{\frac{1}{2}} = 2$	$f(\{0,i\},4)$	$f(\{0,i0_2\},4)$ [5]	
	9	$b^{\frac{1}{2}} = 3$	$f(\{0,1,2\},9)$	$f(\{0,3,6\},9)$	
$D = \frac{1}{3}$	2^3	2	$f(\{0,1\},2^3)$	$f(\{0,2\},2^3)$	$f(\{0,4\},2^3)$
	3^3	3	$f(\{0,1,2\},3^3)$	$f(\{0,3,6\},3^3)$	$f(\{0,9,18\},3^3)$

TABLE 2. Examples of orthonormal bases.
Since the zero is always implicit, it can be omitted from the list of digits.

The first base in Table 2, is a binary decomposition, where the first fracvector represents the binary digits in odd places, and the second fracvector represents the

[5] $i0_2 = i \cdot 2^1 + 0 \cdot 2^0 = 2 \cdot i$

digits in even places, so a decomposition of a number in even and odd digits is unique.

Example: The number $3_{10} = 11_2 = 10_2 + 1_2$

$f_1 = f(\{0,1\},4)$ contains the number $\frac{1}{4}$, and $f_2 = f(\{0,2\},4)$ contains the number $\frac{2}{4}$. By scaling f_1 by $b = 4$, $\frac{1}{4}$ turns into 1, and by scaling f_2 by $b = 4$, $\frac{2}{4}$ turns into 2.

A convenient way to generate the orthonormal basis of fractional dimension $D = \frac{1}{z}$, $z \in \mathbb{Z}$ is to choose a base $b = n^z$, $n \in \mathbb{N}$, select as digits for the first fracversor the smallest consecutive b^D integers $\{0, 1, 2, \ldots, (b^D - 1 = n - 1)\}$, and then for each subsequent fracversor, multiply the selected digits by the number of digits of the first fracvector, $b^D = n$, raised to consecutive powers for each consecutive fracversor. For example, if the first fracversor has n digits, the second fracversor has the same digits, multiplied by n, the third fracversor's digits are multiplied again by n, and so on. This way, the digits of each fracversor are the smallest consecutive multiples of the number of digits of the first fracvector.

This way, each consecutive $\mathbf{m^{th}}$ fracvector is geometrically equal to the first fracvector, scaled by a factor $\mathbf{b^{mD}}$.

Example:

For example, for $D = \frac{1}{4}$ and base $b = 5^4$, a good orthonormal basis with 5 digits, has 4 fracversors:

$f_0 = f(\{0, 1, 2, 3, 4\}, 5^4)$
$f_1 = f(\{0, 5, 10, 15, 20\}, 5^4)$
$f_2 = f(\{0, 25, 50, 75, 100\}, 5^4)$
$f_3 = f(\{0, 125, 250, 375, 500\}, 5^4)$

2.1.1. *Parallelism and orthonormality.*

Definition 4. *Parallelism and orthonormality of fracvectors*

Two fracversors with the same base b, are parallel*, when every digit of one of them is equal to the corresponding digit of the other, except by a factor b^r, $r \in \mathbb{Z}$.*

Two fracversors with the same base b, are orthogonal*, when $\nexists r \in \mathbb{Z}$, $\nexists i \in \mathbb{N} \mid a_i b^r \neq c_j$, for all digits $a_i \neq 0$ and $c_j \neq 0$ of the first and second fracversors respectively.*

2.2. **Geometric product.**

The geometric product of two fracvectors f_x and f_y is the dot product (Inner product, page 9) plus the wedge product (Outer product, page 7):

$$f_x f_y = f_x \cdot f_y + f_x \wedge f_y \tag{2}$$

2.2.0.1. Scaling vs multiplying scalar-product.

In 1-dimensional space, the product of a line segment by a scalar $r \in \mathbb{R}$, produces the same total length as creating r copies of the segment. If we name "multiplication" to the operation of creating copies, we can recognize that in 1-dimensional space, the scaling product produces the same result as the multiplying product. In other dimensions than 1, however, a scaling product is not the same as multiplying product. The difference is measured by the Hausdorff dimension, which is the quotient between the logarithm of the number of copies obtained by the scaling (the multiplication), and the logarithm of the scaling factor that produces that multiplication.

$$D = \frac{\log(\text{multiplication})}{\log(\text{scaling})} \tag{3}$$

As consequence, in general dimension, we need to differentiate between scaling and multiplying products.

In the following, we will call "multiplication" to the classic arithmetical product operation, and "scaling" to the product that distributes between all the base digits of a fracvector. It means that the product of a fracvector with a real number is in general a multiplication, and a scaling by $s \in \mathbb{R}$ is the multiplication by s^D.

Notation:

Multiplication will be denoted by the symbol "$\cdot$", and scaling product by "$\blacksquare$".

Examples:

-Scaling a Sierpinski triangle S[6] by $r = 2$, multiplies it by $r^{\frac{log(3)}{log(2)}} = 3$:

$r \blacksquare S = r^{\frac{log(3)}{log(2)}} \cdot S$

$2 \blacksquare S = 3 \cdot S$

-Scaling a Cantor set C[7] by r, multiplies it by $r^{\frac{log(2)}{log(3)}}$:

$3 \blacksquare C = 2 \cdot C$

-Scaling by r a fracvector of dimension $D = \frac{1}{3}$, like $f(\{0, 1, 2\}, 2^3)$, multiplies it by $r^{\frac{1}{3}}$:

$r \blacksquare f(\{0, 1, 2\}, 2^3) = r^{\frac{1}{3}} \cdot f(\{0, 1, 2\}, 2^3)$

$8 \blacksquare f(\{0, 1, 2\}, 2^3) = 2 \cdot f(\{0, 1, 2\}, 2^3)$

2.2.1. *Outer Product.*

$$A \wedge B = \{a + b \mid a \in A, b \in (B \setminus A)\} \tag{4}$$

The outer product, or wedge product of two fracvectors with the same base, is the Minkowski sum, excluding common elements on both fracvectors: each element of one fracvector is added to each point of the other fracvector that is not present in the first one.

Excluding common digits assures that the Minkowski sum is made between the orthogonal components of the factors.

The dimension of the resulting product has the sum of the dimensions of the orthogonal fracvectors.

$$\begin{aligned} &f(\text{Digits}_A, \mathit{base}) + f(\text{Digits}_B, \mathit{base}) = \\ &= f(\ \{a + b \mid a \in \text{Digits}_A, b \in (\text{Digits}_B \setminus \text{Digits}_A)\}\ , \mathit{base}) \end{aligned} \tag{5}$$

For example, given two fracvectors $A = f(\{0, 1\}, 4)$, and $B = f(\{0, 2\}, 4)$, the outer product is:

$$A \wedge B = f(\{(0 + 0), (1 + 0), (0 + 2), (1 + 2)\}, 4) = f(\{0, 1, 2, 3\}, 4)$$

In this example, A, scaled by powers of the base $b^n = 4^n$, $n \in \mathbb{Z}$ generate the Moser-de Bruijn sequence, OEIS A000695, to which are added fractional values $(n < 0)$.

B generates the sequence A scaled by 2. Both generated sequences are fractals of dimension $D = \frac{1}{2}$, and the wedge product generates all the real numbers, a fractal of dimension $D = 1$.

[6] $S = f(\{0, 1, i\}, 2)$

[7] $C = f(\{0, 2\}, 3)$

In a classic integer dimension, a bivector can be graphically represented as an area, expressed as the wedge product of two $1D$ vectors. Similarly, a $1D$ vector can be expressed as the wedge product of two fracvectors and can be graphically represented in a similar manner. The wedge product can be visualized in $2D$ as a space filling Z-order curve:

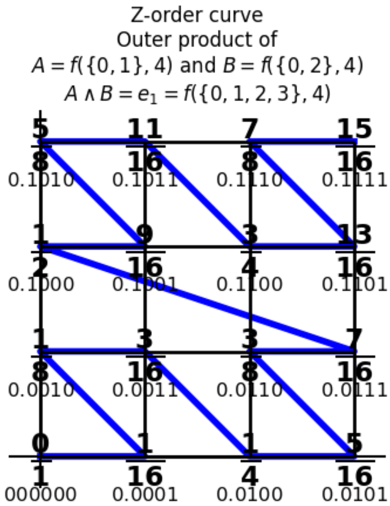

FIGURE 4. Product of fracversors

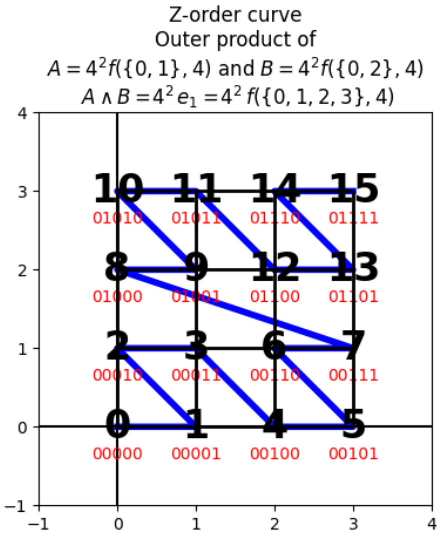

FIGURE 5. Product of fracvectors

FIGURE 6. Z-order curve of the outer product of two fracvectors. To improve clarity, the drawing to the right omits displaying the decimal fractions, but each square should include a similar curve as shown in the entire left figure.

In Figure 6 (a), a $1D$ versor $e_1 = A \wedge B$ is shown as an outer product of two fracversors A, B with dimension $D = \frac{1}{2}$ each one. (Writing e_1 has been omitted from the drawing to simplify the illustration).

The X axis at the bottom represents the first fracversor $A = f(\{0, 1e_1\}, 4)$, and the Y axis at the left represents the second fracversor $B = f(\{0, 2e_1\}, 4)$.

There are infinitely many points on the diagram, but only 16 are depicted, for better visualization, and their binary representations are shown below the decimal numbers. In the graphic, it can be seen how each number (from the set e_1), is the sum of one number from each fracversor (X and Y axis). For example, the number $\frac{3}{8} = 0.0110_2 = 0.12_4$, in e_1, is the sum of $\frac{1}{4} = 0.0100_2 = 0.10_4$ from A, and $\frac{1}{8} = 0.0010_2 = 0.02_4$ from B.[8]

Definition 5 (Anticommutativity of fracversor product)**.** *To extend Clifford algebra* $\mathcal{C}\ell_{n,0,0}$*, the product of orthogonal fracvectors is defined anticommutative:*

[8]Casually, on this case, the binary representation of the numbers in e_1 is equal to the XOR product of the binary representation of points in A and B. This is consequence of A and B not sharing any digit distinct from 0, hence their summation is carry-less, and is equal to a XOR product.

e_1 can be decomposed back in his orthogonal component by doing an AND operation between the binary representations of the numbers in e_1 and the 2-adic representation of $\frac{1}{3} = 0,1111\cdots_4 = 0.010101\cdots_2$. In some sense, $A = e_1 \cap \frac{1}{3}$, $B = e_1 \backslash A$.

$$f_x \wedge f_y = -f_y \wedge f_x$$

2.2.1.1. Outer product of fracvectors to form a basis versor.

A basis versor e_i, in geometric algebra, squaring to 1, is the wedge product of $\frac{1}{D}$ orthonormal fracvectors of dimension D, so the product of the fracvectors should square to 1.

$$e_i = f_D \wedge (b^D \blacksquare f_D) \wedge (b^{2D} \blacksquare f_D) \wedge (b^{3D} \blacksquare f_D) \wedge \cdots \wedge (b^{(\frac{1}{D}-1)D} \blacksquare f_D) \tag{6}$$

For example, for dimension $D = \frac{1}{3}$, e_1 can be expressed as the product of $3 = \frac{1}{D}$ fracversors:

$$f_0 = f(\{0, 1e_1\}, 2^3)$$
$$f_1 = 2 \blacksquare f_0 = f(\{0, 2e_1\}, 2^3)$$
$$f_2 = 2^2 \blacksquare f_0 = f(\{0, 4e_1\}, 2^3)$$

$$e_1 = f_0 \wedge 2 \blacksquare f_0 \wedge 2^2 \blacksquare f_0$$

2.2.2. *Inner Product.*

The inner product of a fracvector with itself, gives the square of his measure, which is the number of copies of the fracversor. If the fracversor uses n digits, each digit counts one equal fraction $\frac{1}{n}$ of the fractal.

2.2.2.1. Measure of scaled versors.

If a fracvector with base b is scaled by b, the result has one copy of the original fractal for each of the b^D digits of the fracversors. The multiplied copies are identical to the original, translated by a vector defined by each one of the base digits.

If it is repeatedly scaled, by multiplying by b^z, $z \in \mathbb{Z}$, the number of copies is b^{Dz}.

Example:

The Sierpinski triangle shown on Figure 2, (page 4), is:

$$f(\{0, 1, i\}, 2) = f(\{\mathbf{0}, \rightarrow, \uparrow\}, 2)$$

If scaled by $b = 2$, it produces $b^D = 3$ translated copies of the original fracversor, each one located at the coordinates of each digit in the base:

$$\mathbf{0} = (0, 0),\ \mathbf{1} = \rightarrow = (1, 0),\ \mathbf{i} = \uparrow = (0, 1)$$

If scaled by b^2, there will be $b^{2D} = 3^2$ translated copies, located at[9]:

$$0, 1, i, 10, 11, 1i, i0, i1, ii$$

But when $z \notin \mathbb{Z}$, the multiplied copies, apart of being translated, also are scaled, and do not match the original size in the fracversor. For that reason we cannot use just a real number to measure a fracvector in general. It only works periodically, when r in the scaling b^r is an integer, so we need to add a complex phase to the measure, with period 1, which is the period of natural numbers between the reals.[10]

[9] Here is used the notation $d_1 d_0 = d_1 \cdot 2^1 + d_0 \cdot 2^0$

[10] $\forall r \in \mathbb{R},\ r \in \mathbb{Z} \rightarrow (r + T \in \mathbb{Z} \leftrightarrow T \equiv 0 \pmod 1)$

Given a fracversor f of dimension D, and measure $\|f\|$, scaling it by a factor b^r, produces a fracvector with measure $\|f\|b^{rD}e^{i\pi r}$. When $r \in \mathbb{Z}$, then $e^{i\pi r} = \pm 1$. [11]

It follows this definition:

Definition 6 (Measure of fracvector)**.**

A fracversor f_D of dimension D, and base b, scaled by c, has a measure:

$$\|c \blacksquare f\| = \|f\|c^D e^{i\pi \log_b(c)} = \|f\|c^{D+\frac{i\pi}{\ln(b)}} \tag{7}$$

- When $c = b^r$ and $r \in \mathbb{Z}$, we have $\|c \blacksquare f\| = \|f\|b^{rD}e^{i\pi r}$, where the complex exponential only affects the sign.
- This complex exponential is chosen to ensure that when the fractional dimension approaches an integer, the sign of the square of the fracversor approaches the sign of the square of a versor with integer dimension, as will be discussed in the subsequent section. This choice is crucial for maintaining sign compatibility and enabling the extension of Clifford algebra to fractional dimensions.
- The logarithm $\log_b(c)$ is the number of digits required to represent c in base b (the lowest significant digit is counted as 0^{th}).
- The measure is signed.

2.2.2.2. Measure of versors.

A 1-blade versor e_i in Clifford algebra $\mathcal{C}\ell_{n,0,0}$ can be decomposed into $\frac{1}{D}$ orthonormal fracversors, each of dimension D.

If $f_D = f(\{0, 1e_i, 2e_i, \cdots (b^D - 1)e_i\}, b)$ is a fracversor of dimension D, by replacing Equation (7) in Equation (6):

$$\|e_i\| = \|f_D\| \cdot \|f_D\|b^{D^2}e^{i\pi(1)} \cdot \|f_D\|b^{2D^2}e^{i\pi(2)} \cdot \ldots \cdot \|f_D\|b^{(\frac{1}{D}-1)D^2}e^{i\pi(\frac{1}{D}-1)} \tag{8}$$

Collecting factors[12]:

$$\|e_i\| = \|f_D\|^{\frac{1}{D}} \cdot b^{\frac{1-D}{2}} \cdot e^{\frac{i\pi(1-D)}{2D}} \tag{9}$$

But in $\mathcal{C}\ell_{n,0,0}$, $\|e_i\| = 1$, then:

$$1 = \|f_D\|^{\frac{1}{D}} \cdot b^{\frac{1-D}{2}} \cdot e^{i\pi\frac{1-D}{2D^2}} \tag{10}$$

Then it follows that:

Definition 7 (Module of Base fracvector)**.**

The module of a base fracvector, $f_D = f(\{0, 1e_1, 2e_1, \cdots, (b^D - 1)e_1\}, b)$ of dimension D, from Equation (10), is:

$$\|f_D\| = b^{\frac{D(D-1)}{2}} \cdot e^{i\pi\frac{D-1}{2D}} \tag{11}$$

Where the purpose of the factor $e^{i\pi\frac{D-1}{2D}}$ is to compensate for the anticommutativity of the fracversor product, as specified in Definition 5, (page 8).

By combining Equation (7) with Equation (11), the module of a scaled base fracvector $c \blacksquare f_D = f(\{0, 1ce_1, 2ce_1, \cdots, (b^D - 1)ce_1\}, b)$ of dimension D is

$$\|c \blacksquare f_D\| = \|f_D\|c^D e^{i\pi \log_b(c)} = b^{\frac{D(D-1)}{2}} \cdot e^{i\pi\frac{D-1}{2D}} c^{D+\frac{i\pi}{\ln(b)}}$$

[11] The imaginary unit used in the measure is a specific imaginary unit, not necessarily the same as the one used elsewhere. When multiplying fracversors by complex numbers, the imaginary units in complex numbers and in the measure, are not assumed to be the same.

[12] Here we make use of the identity $\sum_{i=0}^{n-1} i = \frac{(n-1)n}{2}$

$$\|c \blacksquare f_D\| = c^D \cdot b^{\frac{D(D-1)}{2}} \cdot e^{i\pi\left(\frac{D-1}{2D} + \frac{\ln(c)}{\ln(b)}\right)} \tag{12}$$

In an orthogonal base, of dimension D, for the $f_n = c_n \blacksquare f_0$ fracversor, $c_n = b^{nD}$. Then:

$$\|c_n \blacksquare f_0\| = b^{D\left(\left(n+\frac{1}{2}\right)D-\frac{1}{2}\right)} \cdot e^{i\pi\left(\frac{D-1}{2D}+nD\right)} \tag{13}$$

Where $n \in \{0, 1, \ldots, b^D - 1\}$.

Because the complex exponential is periodic, when $D = \frac{1}{z}$, $z \in \mathbb{N}$ it can be represented as

$$\|c_n \blacksquare f_0\| = b^{D\left(\left(n+\frac{1}{2}\right)D-\frac{1}{2}\right)} \cdot e^{-i\pi\left(\frac{z-1}{2}-\frac{n}{z}\right)} \tag{14}$$

TODO: find formula that mod 2π the imaginary argument.

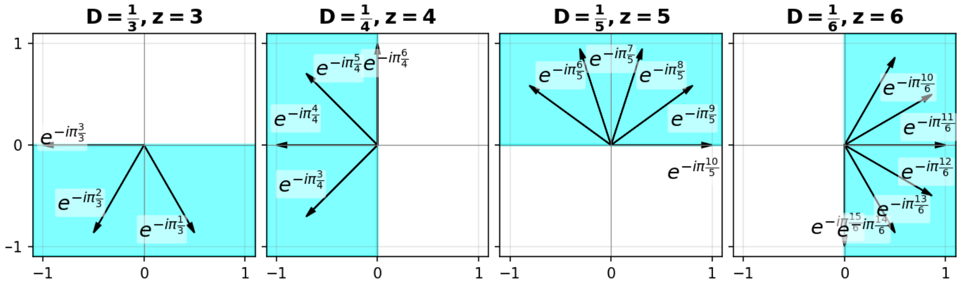

FIGURE 7. Distribution of the complex vector, in the measure of each of the $z = \frac{1}{D}$ fracversors of the orthonormal base, in a half plane, as given by Equation (14).

As shown in Figure 7, Equation (14) shows that the measure of each of the $\frac{1}{D}$ fracversors of the orthogonal base is distributed in a half plane, that rotates $-45°$ for each natural number z in the dimension $D = \frac{1}{z}$. This can be visualized by drawing the measure of the fracversors in the complex plane, and the result will be a spiral that rotates $-45°$ for each natural number z.

Examples:

- The fracvector $f_0 = f(\{0, e_1\}, 4)$ has dimension $D = \frac{1}{2}$, and its measure is:

$$\|f_0\| = 4^{-\frac{1}{8}} \cdot (-i) = \frac{-i}{\sqrt[4]{2}}$$

It is orthogonal to the fracversor $f_1 = 2 \blacksquare f_0 = f(\{0, 2e_1\}, 4)$, which by applying Equation (7), (page 10) has a measure of:

$$\|f_1\| = \|f_0\| 2^{\frac{1}{2}+\frac{i\pi}{\ln(4)}} = \|f_0\| i\sqrt{2} = \sqrt[4]{2}$$

Since the fracversors are orthogonal, $f_0 \cdot f_1 = e_1$, and in $\mathcal{C}\ell_{n,0,0}$, $e_1 e_1 = 1$.

$$e_1^2 = (f_0 f_1)(f_0 f_1) = -(f_0 f_0)(f_1 f_1) = -\left(\frac{-i}{\sqrt[4]{2}}\right)^2 \left(\sqrt[4]{2}\right)^2 = 1$$

Which is the expected result $e_1^2 = 1$. Note that in the calculation, the fracversors anticommute.

c	f_c	$\|f_c\|$
1	$f_0 = f(\{0, e_1\}, 8)$	$\frac{1}{\sqrt[3]{2}} \cdot e^{-i\pi\frac{3}{3}}$
2	$f_1 = f(\{0, 2e_1\}, 8)$	$1 \cdot e^{-i\pi\frac{2}{3}}$
4	$f_2 = f(\{0, 4e_1\}, 8)$	$\sqrt[3]{2} \cdot e^{-i\pi\frac{1}{3}}$

Table 3. Modules of orthogonal fracversors of dimension $D = \frac{1}{3}$, in base $b = 8$.

- For $D = \frac{1}{3}$, base $b = 8$, we have 3 orthogonal fracversors with scales $(c = 1, 2, 4)$, and modules:

$$e_1^2 = (f_0 f_1 f_2)(f_0 f_1 f_2) = -(f_0 f_0)(f_1 f_1)(f_2 f_2) = 1$$

- For $D = \frac{1}{4}$, base $b = 16$, we have 4 orthogonal fracversors with scales $(c = 1, 2, 4, 8)$, and modules:

c	f_c	$\|f_c\|$
1	$f_0 = f(\{0, e_1\}, 16)$	$2^{\frac{-3}{8}} e^{-i\pi\frac{6}{4}}$
2	$f_1 = f(\{0, 2e_1\}, 16)$	$2^{\frac{-1}{8}} e^{-i\pi\frac{5}{4}}$
4	$f_2 = f(\{0, 4e_1\}, 16)$	$2^{\frac{1}{8}} e^{-i\pi\frac{4}{4}}$
8	$f_3 = f(\{0, 8e_1\}, 16)$	$2^{\frac{3}{8}} e^{-i\pi\frac{3}{4}}$

Table 4. Modules of orthogonal fracversors of dimension $D = \frac{1}{4}$, in base $b = 16$.

Unit fracversor of the orthogonal basis

Equation (14) implies that the unit fracversor is the one that makes

$$b^{D\left(\left(n+\frac{1}{2}\right)D-\frac{1}{2}\right)} = 1$$

which corresponds to $D = \frac{1}{2n+1}$, so for $D = \frac{1}{z}$, $n = \frac{z-1}{2}$.

Given that the basis fracversors are enumerated from $n = 0$ to $n = z - 1$, the unit fracversor, in Figure 7, (page 11), is the one just on the middle. In fact, only when z is an odd number, the unit fracversor is in the orthogonal basis.

TODO: maybe the basis should be redefined