Negative rank hyperoperations

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Abstract

We construct the bi-infinite sequence of binary operations where the next element is an iteration of the previous one, such that

$$\dots, +_{r=1}, \times_{r=2}, \hat{}_{r=3}, \dots$$

1 Previous research

When constructing a sequence of binary operations that preserves the addition-multiplication-exponentiation relationship, we must choose which of these properties to construct the sequence based on.

Hyperoperations (Goodstein's hyperoperations)

The hyperoperation sequence $([r])_{r=0}^{\infty}$ is the sequence of binary operations $[r] : \mathbb{N}_0 \times \mathbb{N}_0 \to \mathbb{N}_0$ defined as

$$m[r]n = \begin{cases} n+1 & \text{if } r=0 \\ m & \text{if } r=1, n=0 \\ 0 & \text{if } r=2, n=0 \\ 1 & \text{if } r=3, n=0 \\ m[r-1](m[r](n-1)) & \text{otherwise.} \end{cases}$$

This definition gives the following:

$$m[0]n = n+1, m[1]n = m+n, m[2]n = m\times n, m[3]n = m^n, m[4]n = \underbrace{m^{m^{\cdot^{\cdot^{\cdot^{m}}}}}}_{n \text{ copies of } m}, \dots$$

Note that

$$m[r+1]n = \underbrace{m[r]m[r]\cdots[r]m}_{n \text{ copies of } m} \quad \text{(right-associative)} \tag{1}$$

where $r \geq 1$, but the iteration is one time more than, as

$$m[1]n = \underbrace{m[0]m[0]\cdots[0]m}_{n+1 \text{ copies of } m} \quad \text{(right-associative)}$$

where $r = 0$.

Rubtsov and Trapmann's zeration

There are some attempts to construct a new hyperoperation sequence such that equation (1) holds in general.

Let $☆ : \mathbb{N}_0 \times \mathbb{N}_2 \to \mathbb{N}_0$ be a binary operation.

$$m + n = \underbrace{m☆m☆\cdots☆m}_{n \text{ copies of } m} \quad \text{(right-associative)}$$

if and only if

$$m☆n = \begin{cases} n+2 & \text{if } n = m \\ n+1 & \text{if } n \geq m+2 \quad [1][5]. \end{cases}$$

We consider only where $n \geq 2$ because $m + 1 \neq m$ (1 copy of m). Rubtsov's zeration $[0_R]$ defined as the following fulfills this:

$$m[0_R]n = \begin{cases} n+2 & \text{if } n = m \\ \max(m,n)+1 & \text{if } n \neq m. \end{cases}$$

Rubtsov's zeration is commutative $m[0_R]n = n[0_R]m$, distributive under addition $l + (m[0_R]n) = (l+m)[0_R](l+n)$, and

$$m + n = \underbrace{m[0_R]m[0_R]\cdots[0_R]m}_{n \text{ copies of } m} \quad \text{(left-associative)} \quad [6][7].$$

Also, the exact same operation has been reinvented under the name "junkasan" (which means quasi-addition)[8].

Trapmann's zeration is also a solution of the equation (1)[2]:

$$m[0_T]n = \max(m+2, n+1).$$

This zeration is continuous, different from Rubtsov's one.

There is the attempt to construct a bi-infinite sequence of hyperoperations such that all two adjacent elements satisfy the equation (1)[9].

Lower hyperoperations

Another version of hyperoperation sequence that is applied left-associativity.

Tropical semiring

The binary operation min is commutative, associative, and distributive under addition: $l + \min(m, n) = \min(l + m, l + n)$. We see the finite sequence min, $+, \times$ is an extension of the sequence$+, \times$ that preserves some properties of a commutative ring.

There is the attempt to construct a bi-infinite sequence of binary operations such that all two adjacent elements form a ring[10].

Commutative hyperoperations (Bennett's hyperoperations)

Commutative hyperoperations are the sequence of binary operations defined recursively as

$$m[1]n = m + n, \quad m[r+1]n = e^{(\ln m[r] \ln n)}.$$

This is a generalization of the following relationship between $+$ and $\times$ [11]:

$$m \times n = e^{(\ln m + \ln n)}.$$

The solution is $m[r]n = \exp^{r-1}(\exp^{-(r-1)}(m) + \exp^{-(r-1)}(n))$, and all the next element is distributive over the previous one[3]: $l[r+1](m[r]n) = (l[r+1]m)[r](l[r+1]n)$.

Also, we can easily extend it to bi-infinite sequence. The 0th commutative hyperoperation $m[0]n = \ln(e^m + e^n)$ is distributive under addition[4]: $l + (m[0]n) = (l + m)[0](l + n)$.

Pisa hyperoperations

Pisa hyperoperations are the sequence of binary operations defined recursively as

$$m[2]n = m \times n, \quad m[r+1]n = (m^{\frac{1}{m}})^{(m[r]n)}.$$

This is a generalization of the following relationship between $\times$ and ^:

$$m^n = (m^{\frac{1}{m}})^{(m \hat{} n)}.$$

The $r - 1$th operation is expressed as $m[r-1]n = m \ln_m(m[r]n)$, and we can easily extend it to bi-infinite sequence[12].

Hyperoperations we construct here

We construct the same as Eners49's one[9], the bi-infinite sequence such that $m[1]n = m + n$ and for all $r \in \mathbb{Z}$

$$m[r+1]n = \underbrace{m[r]m[r]\cdots[r]m}_{n \text{ copies of } m} \quad \text{(right-associative)}.$$

From here, consecutive operations are executed from the right unless otherwise stated.

Now, Rubtsov and Trapmann's zerations seem to fit for the 0th element of the bi-infinite hyperoperation sequence. However, a binary operation $\heartsuit$ that meets the following does not exist[7]:

$$m[0_R]n, \text{nor } m[0_T]n = \underbrace{m\heartsuit m\heartsuit\cdots\heartsuit m}_{n \text{ copies of } m}.$$

Proof. Seeking a contradiction, suppose that

$$m[0_R]n = \underbrace{m\heartsuit m\heartsuit\cdots\heartsuit m}_{n \text{ copies of } m} \quad \text{for any } m \in \mathbb{N}_0 \text{ and } n \in \mathbb{N}_2.$$

Then where $m \geq 1$,

$$\begin{aligned} m[0_R](m+1) \quad &= \underbrace{m\heartsuit m\heartsuit\cdots\heartsuit m}_{m+1 \text{ copies of } m} \\ &= m\heartsuit(\underbrace{m\heartsuit m\heartsuit\cdots\heartsuit m}_{m \text{ copies of } m}) \\ &= m\heartsuit(m[0_R]m) \\ &= m\heartsuit(m+2). \end{aligned} \tag{2}$$

Also,

$$\begin{aligned} m[0_R](m+2) \quad &= \underbrace{m\heartsuit m\heartsuit\cdots\heartsuit m}_{m+2 \text{ copies of } m} \\ &= m\heartsuit(\underbrace{m\heartsuit m\heartsuit\cdots\heartsuit m}_{m+1 \text{ copies of } m}) \\ &= m\heartsuit(m[0_R](m+1)) \\ &= m\heartsuit(m+2). \end{aligned} \tag{3}$$

From the equation (2) and (3),

$$m+2 = m[0_R](m+1) = m\heartsuit(m+2) = m[0_R](m+2) = m+3,$$

i.e. $2 = 3$. ☐

We can use the same way to prove that about $[0_T]$. It is a tough question whether there exists such an operation, of which the iteration of its iteration, is equal to addition, while to find what operation repeated n-times results in addition is much easier.

2 The general solution of first order recurrence relations

We now want to know, for given a binary operation $\star$, whether an operation $\heartsuit$ such that

$$\underbrace{m\heartsuit m\heartsuit\cdots\heartsuit m}_{n \text{ copies of } m} = m\star n \quad \text{exists.}$$

Then, if ♡ exists, the sequence

$$m, m☆2, m☆3, m☆4, \ldots \tag{4}$$

is equals to

$$m, m♡m, m♡(m♡m), m♡(m♡(m♡m)), m♡(m♡(m♡(m♡m))), \ldots$$

If we set $a_1 = m, f(n) = m♡n$, then this sequence turns into

$$a_1, f(a_1), f(f(a_1)), f(f(f(a_1))), \ldots$$

That is, whether ♡ exists is equivalent to whether the sequence (4) is defined by a first order recurrence relation (in other words, the sequence is a dynamical system).

From here, Let e be a integer, $\mathbb{N}_e$ be integers not less than e, s be the successor function, and S be a set. Let $P(a, k, k')$ be the statement a is a sequence $\mathbb{N}_e \to S$, $k, k' \in \mathbb{N}_e$, and

$$k < k' \wedge a_k = a_{k'}.$$

$Q(a, p, k)$ be the statement a is a sequence $\mathbb{N}_e \to S$, $p \in \mathbb{N}_1$, $k \in \mathbb{N}_e$,

$$\forall m \forall n; a_{m+k} = a_{pn+m+k} \; (m, n \in \mathbb{N}_0).$$

Lemma 2.1. *Let f be a map defined on S. Suppose a sequence $a : \mathbb{N}_e \to S$ is defined by the first order recurrence relation*

$$a_{n+1} = f(a_n).$$

Then

$$\forall m \in \mathbb{N}_0 \forall k \in \mathbb{N}_e; a_{m+k} = f^m(a_k).$$

Proof. For all $k \in \mathbb{N}_e$, the following holds:

Proof is by induction on m, starting at $m = 0$.

For $m = 0$,

$$f^0(a_n) = a_n = a_{0+n}$$

and so the result holds true for $m = 0$.

Suppose the result holds true for some m, i.e. that $f^m(a_n) = a_{m+n}$. Then we have

$$f^{m+1}(a_n) = f(f^m(a_n)) = f(a_{m+n}) = a_{m+n+1} = a_{(m+1)+n}$$

and so the result holds true for $m+1$ and our proof by induction is complete. □

Lemma 2.2.

a is defined by a first order recurrence relation

$$\Downarrow$$

$$(\, \forall k \forall k'; P(a, k, k') \implies Q(a, k' - k, k) \,).$$

Proof. By the definition of a, there are some map f defined on S such that

$$\forall a_n; a_{n+1} = f(a_n).$$

Let $k, k' \in \mathbb{N}_e$ be

$$k < k' \wedge a_k = a_{k'}.$$

Then we see $k' - k \in \mathbb{N}_0$. By lemma 2.1,

$$\forall k \in \mathbb{N}_e \forall k' \in \mathbb{N}_e; a_k = a_{k'} = a_{k'-k+k} = f^{k'-k}(a_k) \tag{5}$$

For all $m, n \in \mathbb{N}_0$,

$$\begin{aligned} a_{(k'-k)n+m+k} &= f^{(k'-k)n+m}(a_k) && \text{By lemma 2.1} \\ &= f^{(k'-k)n+m}(f^{k'-k}(a_k)) && \text{By (5)} \\ &= f^{(k'-k)n+m+k'-k}(a_k) \\ &= f^{(k'-k)(n+1)+m}(a_k) \\ &= a_{(k'-k)(n+1)+m+k} \end{aligned} \tag{6}$$

Also,

$$a_{(k'-k)0+m+k} = a_{m+k} \tag{7}$$

From (6) and (7),

$$\forall m \forall n; a_{m+k} = a_{(k'-k)n+m+k}.$$

□

Lemma 2.3.

$$Q(a, p_1, k_1) \wedge Q(a, p_2, k_2) \wedge \cdots \implies Q(a, \gcd(p_1, p_2, \dots), \min(k_1, k_2, \dots)).$$

That is

$$Q(a, \gcd(\pi_1[T]), \min(\pi_2[T])),$$

where $T = \{(p, k) | Q(a, p, k)\}$, $\pi_1(t)$ *and* $\pi_2(t)$ *are the first and second coordinate of a pair* t.

Proof. Suppose $p, q \in \mathbb{N}_1$, g is the greatest common divisor of p and q, $k, l \in \mathbb{N}_e$ are $k \leq l$, $Q(a, p, k)$, and $Q(a, q, l)$.

Then we see that for all $m, n \in \mathbb{N}_0$,

$$\begin{aligned} a_{pn+m+l} &= a_{pn+m+l-k+k} \\ &= a_{pn+(m+l-k)+k} \\ &= a_{(m+l-k)+k} && \text{by } Q(a, q, l),\ m + l - k \in \mathbb{N}_0 \\ &= a_{m+l}. \end{aligned} \tag{8}$$

We now check that the linear Diophantine equation

$$px - qy = r$$

has integer solutions (x, y) if and only if r is a multiple of the greatest common divisor of p and q. Then, since $-pq < 0$, it has a positive integer solution.

Moreover, we see $p0 - q0 = g0$. Hence for every $n \in \mathbb{N}_0$, there are $n_1, n_2 \in \mathbb{N}_0$ such that

$$pn_1 = qn_2 + gn. \tag{9}$$

Thus for every $m, n \in \mathbb{N}_0$, there are $n_1, n_2 \in \mathbb{N}_0$ such that

$$\begin{aligned} a_{gn+m+l} &= a_{(gn+m)+l} && \\ &= a_{qn_2+(gn+m)+l} && \text{by } Q(a,q,l),\ gn+m \in \mathbb{N}_0 \\ &= a_{qn_2+gn+m+l} && \\ &= a_{pn_1+m+l} && \text{by (9)} \\ &= a_{m+l} && \text{by (8).} \end{aligned} \tag{10}$$

By the Archimedean property, there is $n_3 \in \mathbb{N}_0$ such that

$$pn_3 + k \geq l \tag{11}$$

Therefore, for every $m, n \in \mathbb{N}_0$, there is $n_3 \in \mathbb{N}_0$ such that

$$\begin{aligned} a_{gn+m+k} &= a_{pn_3+gn+m+k} && \text{by } Q(a,p,k),\ gn+m \in \mathbb{N}_0 \\ &= a_{pn_3+gn+m+k-l+l} && \\ &= a_{gn+(pn_3+m+k-l)+l} && \\ &= a_{(pn_3+m+k-l)+l} && \text{by (10)},\ pn_3+m+k-l \in \mathbb{N}_0 \ (\text{by(11)}) \\ &= a_{pn_3+m+k} && \\ &= a_{m+k} && \text{by } Q(a,p,k). \end{aligned}$$

That is,

$$Q(p_1, k_1) \wedge Q(p_2, k_2) \implies Q(\gcd(p_1, p_2), \min(k_1, k_2)).$$

Therefore

$$Q(a, \gcd(\pi_1[T]), \min(\pi_2[T])),$$

where $T = \{(p,k) | Q(a,p,k)\}$. □

Lemma 2.4.

$$\begin{gathered} (\ \forall k \forall k'; P(a,k,k') \implies Q(a, k'-k, k)\ ) \\ \Downarrow \\ a \textit{ is injective} \vee Q(a, \gcd(\pi_1[T]), \min(\pi_2[T])), \end{gathered}$$

where $T = \{(k'-k, k) \mid P(a,k,k')\}$.

Proof. Where a is injective: The proposition holds.

Where a is not injective: We see $T \neq \varnothing$.

If $Q(a, k'-k, k)$ holds, i.e.

$$\forall m \forall n; a_{m+k} = a_{(k'-k)n+m+k}\ (m, n \in \mathbb{N}_0),$$

then substituting $(m,n) = (0,1)$,

$$a_{0+k} = a_{(k'-k)*1+0+k}, \text{ i.e. } a_k = a_{k'}.$$

Also we see $k'-k \in \mathbb{N}_1$, therefore $k < k'$. $P(a,k,k')$ holds. Hence $\forall k \forall k'; Q(a,k'-k,k) \implies P(a,k,k')$ is true.

From the hypothesis, $\forall k \forall k'; P(a,k,k') \implies Q(a,k'-k,k)$.

Now we have

$$\forall k \forall k'; P(a,k,k') \iff Q(a,k'-k,k).$$

Thus

$$T = \{(k'-k,k) \mid P(a,k,k')\} = \{(k'-k,k) \mid Q(a,k`-k,k)\}.$$

By lemma 2.3,

$$Q(a, \gcd(\pi_1[T]), \min(\pi_2[T])).$$

□

Definition 2.5. *For a sequence $a : \mathbb{N}_e \to S$, define a set $U(a) \subset \mathbb{N}_e$ as*

$$U(a) := \{n \mid \forall m; m \leq n \implies a|_{\{l|l\leq m\}} \textit{ is injective}\}.$$

That is, where $n_1 \in \mathbb{N}_e$ is the maximum integer such that the restriction $a|_{\{l|l\leq n_1\}}$ is injective,

$$U(a) = \begin{cases} \mathbb{N}_e & \textit{if } a \textit{ is injective} \\ \{n \mid n \leq n_1\} & \textit{if } a \textit{ is not injective.} \end{cases}$$

In addition, define a' as a map $a' : U(a) \to a[U(a)]; n \mapsto a_n$.

For example, for a constant sequence $a_n = C$, a' becomes a map $a' : \{e\} \to a[\{e\}]; e \mapsto a_e$ and is bijective. For all sequences a, $U(a)$ exists and $a'$is bijective.

Lemma 2.6. *Suppose a sequence $a : \mathbb{N}_e \to S$ is*

$$a \textit{ is injective} \vee Q(a, g_T, m_T).$$

Then

$$a[U(a)] = a[\mathbb{N}_e],$$

and

$$U(a) = \begin{cases} \mathbb{N}_e & \textit{if } a \textit{ is injective} \\ \{n|n \leq g_T - 1 + m_T\} & \textit{if } a \textit{ is not injective,} \end{cases}$$

where $T = \{(k'-k,k) \mid P(a,k,k')\}$, $g_T = \gcd(\pi_1[T])$, and $m_T = \min(\pi_2[T])$.

Proof. Where a is injective: obvious.

Where a is not injective: We see $Q(a, g_T, m_T)$ holds. Also $T \neq \varnothing$.

Let V be a set $\{n|n \in \mathbb{N}_e, n \leq g_T - 1 + m_T\}$.

Suppose the restriction $a|_V$ is not injective. Then:

There exists $k_1, k_1' \in V$ such that $k_1 < k_1' \wedge a_{k_1} = a_{k_1'}$.

That is, $P(a, k_1, k_1')$ is true.

Therefore $(k_1' - k_1, k_1) \in T$, $k_1' - k_1 \in \pi_1[T]$, $k_1 \in \pi_2[T]$.

For all k_1, $k_1 \leq m_T - 1$ or $k_1 \geq m_T$.

Where $k_1 \leq m_T - 1$:

We have $k_1 \in \pi_2[T]$, contradicting $m_T = \min(\pi_2[T])$.

Where $k_1 \geq m_T$:

$-k_1 \leq -m_T$.
$k_1' \leq g_T - 1 + m_T$ by $k_1' \in V$.
Therefore $k_1' - k_1 \leq g_T - 1 + m_T - m_T = g_T - 1$.
We see $k_1' - k_1 \in \pi_1[T]$, contradicting $g_T = \gcd(\pi_1[T])$.

Thus the assumption $a|_V$ is not injective leads to a contradiction.
Hence $a|_V$ is injective.

Let V' be a set $\{n | n \in \mathbb{N}_e, n \leq g_T + m_T\}$. Then we have $m_T, g_T + m_T \in V'$. By $Q(a, g_T, m_T)$, $a_{m_T} = a_{g_T + m_T}$. Therefore

$$m_T \neq g_T + m_T \land a_{m_T} = a_{g_T + m_T}.$$

The restriction $a|_{V'}$ is not injective.
By definition 2.5,

$$U(a) = V = \{n | n \in \mathbb{N}_e, n \leq g_T - 1 + m_T\}.$$

For all $a_n \in a[\mathbb{N}_e]$, $n \leq m_T - 1$ or $n \geq m_T$.
Where $n \leq m_T - 1$:
By $n \in U(a)$, $a_n \in a[U(a)]$.
Where $n \geq m_T$:
There exists $n_1, n_2 \in \mathbb{N}_0$ such that

$$n = g_T n_1 + n_2 + m_T \land n_2 \leq g_T - 1.$$

By $n_2 \leq g_T - 1$, $n_2 + m_T \leq g_T - 1 + m_T$.

$n_2 + m_T \in U(a)$.

From $Q(a, g_T, m_T)$, $a_n = a_{g_T n_1 + n_2 + m_T} = a_{n_2 + m_T} \in a[U(a)]$.
Thus for all $a_n \in a[\mathbb{N}_e]$, $a_n \in a[U(a)]$.
$a[\mathbb{N}_e] \subset a[U(a)]$. Additionally, By $U(a) \subset \mathbb{N}_e$, $a[U(a)] \subset a[\mathbb{N}_e]$. Hence $a[U(a)] = a[\mathbb{N}_e]$. □

Lemma 2.7.

$$a \textit{ is injective} \lor Q(a, g_T, m_T) \implies a \textit{ is defined by a first recurrence relation},$$

where $T = \{(k' - k, k) \mid P(a, k, k')\}$, $g_T = \gcd(\pi_1[T])$, *and* $m_T = \min(\pi_2[T])$.

Proof. By lemma 2.6, $a[U(a)] = a[\mathbb{N}_e]$.

For all $a_n \in a[\mathbb{N}_e]$, There exists $n_1 \in U(a)$ such that $a_n = a_{n_1}$.

$$a_{n+1} = a_{n_1+1} = (a \circ s)(n_1) = (a \circ s \circ a^{!-1} \circ a^{!})(n_1)$$

$$= (a \circ s \circ a^{!-1})(a^{!}(n_1)) = (a \circ s \circ a^{!-1})(a_{n_1}) = (a \circ s \circ a^{!-1})(a_n).$$

The sequence a is defined by the recurrence relation

$$a_{n+1} = (a \circ s \circ a^{!-1})(a_n).$$

□

By lemma 2.2, 2.4, and 2.7,

$$\begin{gathered} a \text{ is defined by a first recurrence relation} \\ \Downarrow \\ \forall k \forall k'; P(a,k,k') \implies Q(a,k'-k,k) \\ \Downarrow \\ a \text{ is injective} \vee Q(a, g_T, m_T) \\ \Downarrow \\ a \text{ is defined by a first recurrence relation} \end{gathered}$$

These three propositions are equivalent.

Lemma 2.8. *Suppose a sequence $a : \mathbb{N}_e \to S$ is defined by a first order recurrence relation. Let f be a map defined on S. Then*

$$\forall a_n \in a[\mathbb{N}_e]; a_{n+1} = f(a_n) \iff \forall x \in a[\mathbb{N}_e]; f(x) = a \circ s \circ a^{!-1}(x).$$

Proof. **Proof of ($\implies$):** By lemma 2.6, $a[U(a)] = a[\mathbb{N}_e]$. For all $x \in a[\mathbb{N}_e]$, $x \in a[U(a)]$. There exists $n_1 \in U(a)$ such that $x = a_{n_1}$.

$$f(x) = f(a_{n_1}) = a_{n_1+1} = (a \circ s)(n_1) = (a \circ s \circ a^{!-1} \circ a^{!})(n_1)$$

$$= (a \circ s \circ a^{!-1})(a^{!}(n_1)) = (a \circ s \circ a^{!-1})(a_{n_1}) = (a \circ s \circ a^{!-1})(x).$$

(Proof of $\impliedby$): By lemma 2.6, $a[U(a)] = a[\mathbb{N}_e]$. For all $a_n \in a[\mathbb{N}_e]$, There exists $n_1 \in U(a)$ such that $a_n = a_{n_1}$.

$$f(a_n) = (a \circ s \circ a^{!-1})(a_n) = (a \circ s \circ a^{!-1})(a_{n_1}) = (a \circ s \circ a^{!-1})(a^{!}(n_1))$$

$$= (a \circ s \circ a^{!-1} \circ a^{!})(n_1) = (a \circ s)(n_1) = (a \circ s)(n) = a_{n+1}.$$

□

If we replace "a is a sequence defined by a first order recurrence relation" in lemma 2.8 with simply "a is a sequence", then the proposition becomes false. Counterexample: the sequence

$$1, 2, 1, 1, 1, 1, \ldots$$

3 Definition of *multiplicative* and *additive*

We now decide the domain and the codomain of operations we deal with in this article. Suppose binary operations $\star, \heartsuit : \mathbb{N}_0 \times \mathbb{N}_0 \to \mathbb{N}_0$, which does not necessarily form a ring, satisfy

$$\underbrace{m\star m\star \cdots \star m}_{n \text{ copies of } m} = m\heartsuit n.$$

Then $m\heartsuit 1 = m$ (1 copy of m) for all m, no matter what operation $\heartsuit$ is. Addition does not suit for the operation $\heartsuit$. Furthermore, let $♪ : \mathbb{N}_0 \times \mathbb{N}_0 \to \mathbb{N}_0$ satisfy

$$\underbrace{m\heartsuit m\heartsuit \cdots \heartsuit m}_{n \text{ copies of } m} = m ♪ n.$$

Then

$$1 ♪ n = \underbrace{1\heartsuit 1\heartsuit \cdots \heartsuit 1}_{n \text{ copies of } 1} = 1 \text{ for all } n.$$

Even multiplication does not suit for $♪$. Thus we only deal with operations whose domain is the ordered pair of integers not less than 2. Also, because the value of

$$\underbrace{m\star m\star \cdots \star m}_{n \text{ copies of } m}$$

can be undefined where $n \geq 3$, we decide the codomain is integers not less than 2.

Definition 3.1. *Multiplicative of* $\star$*: For a binary operation* $\star : \mathbb{N}_2 \times \mathbb{N}_2 \to \mathbb{N}_2$ *and* $m, n \in \mathbb{N}_2$*, define* $m\star^{(1)}n$ *as follows:*

$$m\star^{(1)}n := \underbrace{m\star m\star \cdots \star m}_{n \text{ copies of } m} = \begin{cases} m\star m & \text{if } n = 2 \\ m\star(m\star^{(1)}(n-1)) & \text{if } n \geq 3. \end{cases}$$

Then $(m, n) \mapsto m\star^{(1)}n$ *is a binary operation defined on* $\mathbb{N}_2 \times \mathbb{N}_2$*. Obviously* $m\star^{(1)}n \in \mathbb{N}_2$ *for all* $m, n \in \mathbb{N}_2$*. We say that the map* $\star^{(1)} : \mathbb{N}_2 \times \mathbb{N}_2 \to \mathbb{N}_2$ *is the multiplicative of* $\star$*. Uniqueness of* $\star^{(1)}$ *is clear.*

We call $\star^{\overbrace{(1)(1)\dots(1)}^{n \text{ copies of } (1)}}$ *the nth multiplicative of* $\star$*, and denote* $\star^{(n)}$*. Also,* $\star^{(0)} := \star$*.*

Definition 3.2. *Additive of* $\star$*: For a binary operation* $\star : \mathbb{N}_2 \times \mathbb{N}_2 \to \mathbb{N}_2$*, a binary operation* $\star^{(-1)} : \mathbb{N}_2 \times \mathbb{N}_2 \to \mathbb{N}_2$ *is said to be an additive of* $\star$ *if* $\star^{(-1)(1)} = \star$*.*

We call $\star^{\overbrace{(-1)(-1)\dots(-1)}^{n \text{ copies of } (-1)}}$ *a nth additive of* $\star$*, and denote* $\star^{(-n)}$*.*

The additive of $\star$, $\star^{(-1)}$ does not always represent a specific operation like the indefinite integral $\int f(x)dx$. A range of $\star$ such that $(\star, n) \mapsto \star^{(-n)}$ is a well-defined map appears in a later section.

For a binary operation $\star$, there is no relation between whether the operation $\heartsuit$ satisfy the following exists:

$$a \star n = \underbrace{a \heartsuit a \heartsuit \cdots \heartsuit a}_{n \text{ copies of } a} \text{ for all } n \in \mathbb{N}_2,$$

and the value of $b \star n$ $(b \neq a)$. For example, let $m \star n = 2m + n - 4$. Then the operation $\heartsuit$ such that

$$2 \star n = \underbrace{2 \heartsuit 2 \heartsuit \cdots \heartsuit 2}_{n \text{ copies of } 2}$$

does not exist, and where $m \geq 3$,

$$m \star n = \underbrace{m \heartsuit m \heartsuit \cdots \heartsuit m}_{n \text{ copies of } m}$$

for some operations $\heartsuit$. We will ask the existence of such operation $\heartsuit$ for each m.

Let a sequence $a, b : \mathbb{N}_2 \to \mathbb{N}_2$ satisfy

$$a_n = m \star n, \quad b_n = m \heartsuit n, \quad m \star n = \underbrace{m \heartsuit m \heartsuit \cdots \heartsuit m}_{n \text{ copies of } m}.$$

Then

$$a_n = \begin{cases} b_m & \text{if } n = 2 \\ b_{a_{n-1}} & \text{if } n \geq 3. \end{cases}$$

Define $a_1 := m$, then

$$a_n = \begin{cases} m & \text{if } n = 1 \\ b_{a_{n-1}} & \text{if } n \geq 2. \end{cases}$$

We now have $a_n = b_{a_{n-1}}$ for all $n \in \mathbb{N}_2$, the original domain of a. Also, define $a_0 := 1, b_1 := m$, then

$$a_n = \begin{cases} 1 & \text{if } n = 0 \\ b_{a_{n-1}} & \text{if } n \geq 1. \end{cases}$$

Define $a_{-1} := 0, b_0 := 1$, then

$$a_n = \begin{cases} 0 & \text{if } n = -1 \\ b_{a_{n-1}} & \text{if } n \geq 0. \end{cases}$$

Define $a_{-2} := -1, b_{-1} = 0$, then...

Now we show that the existence of $\heartsuit$ s.t.

$$m\star n = \underbrace{m\heartsuit m\heartsuit\cdots\heartsuit m}_{n \text{ copies of } m}$$

is equivalent to the existence of B s.t.

$$A_n = B_{A_{n-1}},$$

where bi-infinite sequence $A, B : \mathbb{Z} \to \mathbb{Z}$ is defined as

$$A_n = \begin{cases} n+1 & \text{if } n \le 0 \\ m & \text{if } n = 1 \\ m\star n & \text{if } n \ge 2, \end{cases} \qquad B_n = \begin{cases} n+1 & \text{if } n \le 0 \\ m & \text{if } n = 1 \\ m\heartsuit n & \text{if } n \ge 2. \end{cases}$$

From here, let $P(A, e)$ be the statement A is a bi-infinite sequence $\mathbb{Z} \to \mathbb{Z}$, $A_n = n+1$ where $n \le e$, and $A_n \ge e+2$ where $n \ge e+1$.

Definition 3.3. *Multiplicative of A: For A s.t. $P(A, 0)$, the bi-infinite sequence $A^{(1)} : \mathbb{Z} \to \mathbb{Z}$ is said to be the multiplicative of A if*

$$\forall n \in \mathbb{Z}; A_n^{(1)} = A_{A_{n-1}^{(1)}} \wedge A_0^{(1)} = A_0.$$

It is structurally the same thing whatever integer the 0 is rewritten as.

Generally $P(A^{(1)}, 0)$ holds true.

Proof. Proof is by induction on n. For $n = 0$:

$$A_0^{(1)} = A_0 = 0+1 \text{ by definition 3.3.}$$

Suppose that $A_n^{(1)} = n+1$ for some $n \le 0$. Then obviously $A_{n-1}^{(1)} \le 0$ because $n-1 \le -1$.

$$\begin{aligned} n+1 &= A_n^{(1)} && \text{by the hypothesis} \\ &= A_{A_{n-1}^{(1)}} && \text{by defnition 3.3} \\ &= A_{n-1}^{(1)} + 1 && \because A_{n-1}^{(1)} \le 0, \end{aligned}$$

i.e. $A_{n-1}^{(1)} = (n-1)+1$. By mathematical induction $A_n^{(1)} = n+1$ for any $n \le 0$.

And for $n = 1$:

$$A_1^{(1)} = A_{A_0^{(1)}} = A_1 \ge 2 \text{ by definition 3.3.}$$

Suppose that $A_n^{(1)} \ge 2$ for some $n \ge 1$. Then

$$A_{n+1}^{(1)} = A_{A_n^{(1)}} \ge 2 \text{ by definition 3.3.}$$

By mathematical induction $A_n^{(1)} \ge 2$ for any $n \ge 1$. $\square$

Because $P(A^{(1)}, 0)$ holds true, $A^{(1)}$ is unique. A map$(A, n) \mapsto A^{(n)}$ is well-defined.

Definition 3.4. *Additive of A: For A s.t. $P(A, 0)$, the bi-infinite sequence $A^{(-1)} : \mathbb{Z} \to \mathbb{Z}$ is said to be an additive of A if*

$$P(A^{(-1)}, 0) \wedge (\ \forall n \in \mathbb{Z}; A_n = A^{(-1)}_{A_{n-1}}\ ) \wedge A^{(-1)}_0 = A_0,$$

that is

$$P(A^{(-1)}, 0) \wedge (A^{(-1)})^{(1)} = A.$$

We included the condition that $P(A^{(-1)}, 0)$ holds true in the definition, because there are A, B such that

$$P(A, 0) \wedge \neg P(B, 0) \wedge (\ \forall n \in \mathbb{Z}; A_n = B_{A_{n-1}}\ ) \wedge B_0 = A_0,$$

e.g.

$$A_n = \begin{cases} n+1 & \textit{if } n \leq 0 \\ 5 & \textit{if } n = 1 \\ n+5 & \textit{if } n \geq 2, \end{cases} \quad B_n = \begin{cases} n+1 & \textit{if } n \leq 0 \\ 5 & \textit{if } n = 1 \\ -1 & \textit{if } 2 \leq n \leq 4 \\ 7 & \textit{if } n = 5 \\ -1 & \textit{if } n = 6 \\ n+1 & \textit{if } n \geq 7. \end{cases}$$

Similar to the binary operation's ones, we call $A^{\overbrace{(1)(1)\ldots(1)}^{n \text{ copies of } (1)}}$ the nth *multiplicative* of A, denote $A^{(n)}$, and vice versa. Also, we define $A^{(0)} := A$.

4 Properties of *multiplicative* and *additive*

Lemma 4.1. *Let A be the bi-infinite sequence s.t. $P(A, 0)$. If A is bijective, $A^{(-1)}$ exists and is uniquely determined as*

$$A^{(-1)} = A \circ s \circ A^{-1}.$$

Moreover, $A^{(-1)}$ is bijective.

Proof. **(Existence)**: A sequence $a : \mathbb{N}_1 \to \mathbb{N}_2; n \mapsto A_n$ is bijective because of $P(A, 0)$ and that A is bijective. By lemma 2.7, a is defined by a first order recurrence relation, so there exists a map $f : \mathbb{N}_2 \to \mathbb{N}_2$ where

$$a_{n+1} = f(a_n).$$

Define a bi-infinite sequence $B : \mathbb{Z} \to \mathbb{Z}$ as

$$B_n := \begin{cases} n+1 & \text{if } n \leq 0 \\ a_1 & \text{if } n = 1 \\ f(n) & \text{if } n \geq 2. \end{cases}$$

Then clearly $P(B, 0)$ holds true.
Where $n \leq 0$:

$A_n = n + 1 = B_n = B_{A_{n-1}}$.
Where $n = 1$:
$A_1 = a_1 = B_1 = B_{A_0}$.
Where $n \geq 2$:
$A_n = a_n = f(a_{n-1}) = B_{a_{n-1}} = B_{A_{n-1}} \because a_{n-1} \geq 2$.
Thus B is one of *additive* of A, $A^{(-1)}$ by definition 3.4.

(Uniqueness): Let $C : \mathbb{Z} \to \mathbb{Z}$ be an *additive* of A.
Where $n \leq 0$:
A is bijective, and $A(n) = n + 1$ if $n \leq 0$;

therefore $A^{-1}(n) = n - 1$ if $n \leq 1$.

Thus $C(n) = n + 1 = n + 1 + 1 - 1 = (A \circ s \circ A^{-1})(n) \because s(A^{-1}(n)) \leq 0$.
Where $n = 1$:
$C(1) = C(A(0)) = A(1) = (A \circ s)(0) = (A \circ s \circ A^{-1})(1)$.
Where $n \geq 2$:
Let a be a sequence $\mathbb{N}_1 \to \mathbb{N}_2; n \mapsto A(n)$.

From C is an *additive* of A, $A(n + 1) = C(A(n))$ for all $n \in \mathbb{N}_1 \subset \mathbb{Z}$.

That is $a(n + 1) = C(a(n))$ for all $n \in \mathbb{N}_1$.

By lemma 2.8, $C(n) = (a \circ s \circ a^{!-1})(n)$ for all $n \in a[\mathbb{N}_1] = \mathbb{N}_2$.

A sequence $a : \mathbb{N}_1 \to \mathbb{N}_2; n \mapsto A_n$ is bijective from the assumption.

By definition 2.5, $U(a) = \mathbb{N}_2$ and $a^! = a$.

$A(A^{-1}(n)) = n = a(a^{-1}(n)) = A(a^{-1}(n))$ for all $n \in \mathbb{N}_2$.

In addition, A is bijective.

Therefore $A^{-1}(n) = a^{-1}(n)$ for all $n \in \mathbb{N}_2$.

$C(n) = (a \circ s \circ a^{!-1})(n) = (a \circ s \circ a^{-1})(n) = (A \circ s \circ A^{-1})(n)$ for all $n \in \mathbb{N}_2$.
Thus C, the *additive* of A uniquely determined as $C = A \circ s \circ A^{-1}$. In other words $A^{(-1)}$ is well-defined where A is bijective.

(Bijectiveness): Since A, A^{-1}, and the successor function $s : \mathbb{Z} \to \mathbb{Z}$ is bijective, the composite function $A \circ s \circ A^{-1} = A^{(-1)}$ is bijective.

Because of this, $(A, n) \mapsto A^{(-n)}$ is a well-defined map. □

By the way, an *additive* of A does not exist where A is surjective and not injective.

Proof. Seeking a contradiction, suppose $A^{(-1)}$ exists. Then:
Let a be a sequence $\mathbb{N}_1 \to \mathbb{N}_2; n \mapsto A(n)$.

a is defined by a first order recurrence that $a(n + 1) = A^{(-1)}(a(n))$.

By lemma 2.6, $a[U(a)] = a[\mathbb{N}_1]$.

Because of $P(A, 0)$, $a : \mathbb{N}_1 \to \mathbb{N}_2$ is surjective and not injective.

Since a is not injective, by definition 2.5 $U(a)$ is finite.

Thus $a[U(a)]$ is finite.

Meanwhile, since a is surjective, $a[\mathbb{N}_1] = \mathbb{N}_2$ is infinite.

We now arrive at a contradiction.

□

Lemma 4.2. *Suppose a binary operator $☆ : \mathbb{N}_2 \times \mathbb{N}_2 \to \mathbb{N}_2$, a bi-infinite sequence $A : \mathbb{Z} \to \mathbb{Z}$, and an integer $m \geq 2$ satisfy that*

$$\forall n; A(n) = \begin{cases} n+1 & \text{if } n \leq 0 \\ m & \text{if } n = 1 \\ m☆n & \text{if } n \geq 2. \end{cases}$$

Then

$$\forall n; A^{(1)}(n) = \begin{cases} n+1 & \text{if } n \leq 0 \\ m & \text{if } n = 1 \\ m☆^{(1)}n & \text{if } n \geq 2. \end{cases}$$

Proof. Because $m \geq 2$ and $\forall n; m☆n \geq 2$, $P(A, 0)$ holds true. Thus $A^{(1)} : \mathbb{Z} \to \mathbb{Z}$ is defined.
Where $n \leq 0$:
$A^{(1)}(n) = n + 1$ because $P(A^{(1)}, 0)$ holds true.
Where $n = 1$:
$A^{(1)}(1) = A(A^{(1)}(0)) = A(1) = m$.
Where $n \geq 2$:
$m☆^{(1)}n \geq 2$.

$A^{(1)}(2) = A(A^{(1)}(1)) = A(m) = m☆m = m☆^{(1)}2$.

Suppose $A^{(1)}(n) = m☆^{(1)}n$ for some n. Then:

$$A^{(1)}(n+1) = A(A^{(1)}(n)) = A(m☆^{(1)}n) = m☆(m☆^{(1)}n) = m☆^{(1)}(n+1).$$

Therefore, by mathematical induction $\forall n; A^{(1)}(n) = m☆^{(1)}n$.

□

Lemma 4.3. *Suppose a binary operator* $\text{☆} : \mathbb{N}_2 \times \mathbb{N}_2 \to \mathbb{N}_2$*, a bi-infinite sequence* $A : \mathbb{Z} \to \mathbb{Z}$*, and an integer* $m \geq 2$ *satisfy that*

$$\forall n \in \mathbb{Z}; A(n) = \begin{cases} n+1 & \textit{if } n \leq 0 \\ m & \textit{if } n = 1 \\ m\text{☆}n & \textit{if } n \geq 2. \end{cases}$$

Let A *be bijective. Then a binary operator* $\heartsuit : \mathbb{N}_2 \times \mathbb{N}_2 \to \mathbb{N}_2$ *such that*

$$\forall n \in \mathbb{N}_2; m\text{☆}n = \begin{cases} m\heartsuit m & \textit{if } n = 2 \\ m\heartsuit(m\text{☆}(n-1)) & \textit{if } n \geq 3 \end{cases}$$

exists and uniquely determined as

$$\forall n \in \mathbb{N}_2; m\heartsuit n = (A \circ s \circ A^{-1})(n).$$

That is,

$$\forall n \in \mathbb{Z}; A^{(-1)}(n) = \begin{cases} n+1 & \textit{if } n \leq 0 \\ m & \textit{if } n = 1 \\ m\heartsuit n & \textit{if } n \geq 2. \end{cases}$$

Proof. **(Existence)**: By lemma 4.1, $A^{(-1)}$ exists. Let a binary operation $\text{♪} : \mathbb{N}_2 \times \mathbb{N}_2 \to \mathbb{N}_2$ be

$$\forall n \in \mathbb{N}_2; m \,\text{♪}\, n = A^{(-1)}(n).$$

By definition 3.4,

$$m\text{☆}2 = A(2) = A^{(-1)}(A(1)) = A^{(-1)}(m) = m \,\text{♪}\, m$$

and

$$\forall n \geq 3; m\text{☆}n = A(n) = A^{(-1)}(A(n-1)) = A^{(-1)}(m\text{☆}(n-1)) = m \,\text{♪}\, ((m\text{☆}(n-1)).$$

(Uniqueness): Suppose a binary operation $\text{❄} : \mathbb{N}_2 \times \mathbb{N}_2 \to \mathbb{N}_2$ fulfills

$$\forall n \in \mathbb{N}_2; m\text{☆}n = \begin{cases} m\text{❄}m & \text{if } n = 2 \\ m\text{❄}(m\text{☆}(n-1)) & \text{if } n \geq 3. \end{cases}$$

Define a bi-infinite sequence $C : \mathbb{Z} \to \mathbb{Z}$ as

$$\forall n \in \mathbb{Z}; C(n) := \begin{cases} n+1 & \text{if } n \leq 0 \\ m & \text{if } n = 1 \\ m\text{❄}n & \text{if } n \geq 2. \end{cases}$$

Then clearly $P(C, 0)$ holds true.
Where $n \leq 0$:
$A(n) = n + 1 = C(n) = C(A(n-1))$.

Where $n = 1$:
$A(1) = m = C(1) = C(A(0))$.
Where $n = 2$:
$A(2) = m☆2 = m❊m = C(m) = C(A(1))$.
Where $n \geq 3$:
$m☆(n-1) \geq 2$.

$A(n) = m☆n = m❊(m☆(n-1)) = m❊(A(n-1)) = C(A(n-1))$.
Thus C is the *additive* of A.
By lemma 4.1, C is uniquely determined as $A \circ s \circ A^{-1}$.

$$\forall n \in \mathbb{N}_2; m❊n = C(n) = (A \circ s \circ A^{-1})(n).$$

In other words $☆^{(-1)}$ is well-defined if $\forall m \in \mathbb{N}_2; A : \mathbb{Z} \to \mathbb{Z}$ is bijective, where

$$\forall n \in \mathbb{Z}; A(n) = \begin{cases} n+1 & \text{if } n \leq 0 \\ m & \text{if } n = 1 \\ m☆n & \text{if } n \geq 2. \end{cases}.$$

□

5 The negative hyperoperations

This proposition 5.1 was proved at [1] and [5]. However we try to prove again using lemmas we have proved here.

Proposition 5.1. *Let* $☆$ *be a binary operation* $\mathbb{N}_2 \times \mathbb{N}_2 \to \mathbb{N}_2$.

$$☆^{(1)} = + : \mathbb{N}_2 \times \mathbb{N}_2 \to \mathbb{N}_2 \iff$$

$$\forall m \in \mathbb{N}_2 \forall n \in \{m\} \cup \mathbb{Z}_{\geqslant m+2}; m☆n = \begin{cases} m+2 & \textit{if } n = m \\ n+1 & \textit{if } n \geq m+2 \end{cases}$$

Proof. **(Proof of $\implies$)**: For all $m \in \mathbb{N}_2$, the following holds:
Define a sequence $a : \mathbb{N}_1 \to \mathbb{N}_2$ as

$$\forall n \in \mathbb{N}_1; a(n) := \begin{cases} m & \text{if } n = 1 \\ m+n & \text{if } n \geq 2. \end{cases}$$

From the assumption,

$$\forall n \in \mathbb{N}_2; m+n = \begin{cases} m☆m & \text{if } n = 2 \\ m☆(m+(n-1)) & \text{if } n \geq 3. \end{cases}$$

Therefore

$$\forall n \in \mathbb{N}_1; a(n+1) = m☆(a(n)).$$

By lemma 2.8,

$$\forall n \in a[\mathbb{N}_1]; m \text{☆} n = (a \circ s \circ a^{!-1})(n).$$

Clearly $a[\mathbb{N}_1] = \{m\} \cup \mathbb{Z}_{\geqslant m+2}$. By definition 2.5, $U(a) = \mathbb{N}_1$. By lemma 2.6, $a[U(a)] = a[\mathbb{N}_1] = \{m\} \cup \mathbb{Z}_{\geqslant m+2}$. The map $a^{!-1} : \{m\} \cup \mathbb{Z}_{\geqslant m+2} \to \mathbb{N}_1$ is determined as

$$\forall n \in \{m\} \cup \mathbb{Z}_{\geqslant m+2}; a^{!-1}(n) = \begin{cases} 1 & \text{if } n = m \\ n - m & \text{if } n \geq m + 2. \end{cases}$$

Thus

$$\forall n \in \{m\} \cup \mathbb{Z}_{\geqslant m+2}; m \text{☆} n = (a \circ s \circ a^{!-1})(n) = \begin{cases} (a \circ s \circ a^{!-1})(n) & \text{if } n = m \\ (a \circ s \circ a^{!-1})(n) & \text{if } n \geq m + 2 \end{cases}$$

$$= \begin{cases} (a \circ s)(1) & \text{if } n = m \\ (a \circ s)(n - m) & \text{if } n \geq m + 2 \end{cases} = \begin{cases} a(2) & \text{if } n = m \\ a(n - m + 1) & \text{if } n \geq m + 2 \end{cases}$$

$$= \begin{cases} m + 2 & \text{if } n = m \\ m + (n - m + 1) & \text{if } n \geq m + 2 \end{cases} (\because n - m + 1 \geq 2) = \begin{cases} m + 2 & \text{if } n = m \\ n + 1 & \text{if } n \geq m + 2. \end{cases}$$

(Proof of $\Longleftarrow$): For all $m \in \mathbb{N}_2$, the following holds:
From the assumption,

$$m + 2 = m \text{☆} m.$$

If $n \geq 3$, then $m + (n - 1) \geq m + 2$. Therefore

$$\forall n \geq 3; m + n = (m + (n - 1)) + 1 = m \text{☆} (m + (n - 1)).$$

Thus

$$\forall m \in \mathbb{N}_2 \forall n \in \mathbb{N}_2; m + n = \begin{cases} m \text{☆} m & \text{if } n = 2 \\ m \text{☆} (m + (n - 1)) & \text{if } n \geq 3, \end{cases}$$

i.e. $\text{☆}^{(1)} = +$. □

Proposition 5.2. *Define a binary operation* $\heartsuit : \mathbb{N}_2 \times \mathbb{N}_2 \to \mathbb{N}_2$ *as*

$$m \heartsuit n := \begin{cases} n & \text{if } 2 \leq n \leq m - 1 \\ m + 2 & \text{if } n = m \\ m + 1 & \text{if } n = m + 1 \\ n + 1 & \text{if } n \geq m + 2. \end{cases}$$

Then for all $m, n \in \mathbb{N}_2$*, and* $r \in \mathbb{Z}$*,* $m \heartsuit^{(r)} n$ *exists and uniquely determined.*

Proof. For all $m \in \mathbb{N}_2$, the following holds:

Define a bi-infinite sequence $A : \mathbb{Z} \to \mathbb{Z}$ as

$$\forall n \in \mathbb{Z}; A(n) := \begin{cases} n+1 & \text{if } n \leq 0 \\ m & \text{if } n = 1 \\ m \heartsuit n & \text{if } n \geq 2. \end{cases}$$

Where $m = 2$:

$$\forall n \in \mathbb{Z}; A(n) = \begin{cases} n+1 & \text{if } n \leq 0 \\ 2 & \text{if } n = 1 \\ 4 & \text{if } n = 2 \\ 3 & \text{if } n = 3 \\ n+1 & \text{if } n \geq 4, \end{cases}$$

i.e.

$$\ldots, \quad -1, \quad 0, \quad 1, \quad 2, \quad 4, \quad 3, \quad 5, \quad 6, \quad 7, \quad \ldots$$

Where $m \geq 3$:

$$\forall n \in \mathbb{Z}; A(n) = \begin{cases} n+1 & \text{if } n \leq 0 \\ m & \text{if } n = 1 \\ n & \text{if } 2 \leq n \leq m-1 \\ m+2 & \text{if } n = m \\ m+1 & \text{if } n = m+1 \\ n+1 & \text{if } n \geq m+2, \end{cases}$$

i.e.

$$\ldots, \quad -1, \quad 0, \quad 1, \quad m, \quad \underbrace{2, \quad \ldots, \quad m-1,}_{\text{Where } 2 \leq n \leq m-1} \quad m+2, \quad m+1, \quad m+3, \quad m+4, \quad \ldots$$

Obviously A is bijective for any cases.

By definition 3.3, for all $r \geq 0$ there exists a unique $A^{(r)}$.

Futhermore, $A = A^{(0)}$ is uniquely defined and bijective. Suppose there exists a unique $A^{(r)}$ and the $A^{(r)}$ is bijective for some $r \leq 0$. Then, by lemma 4.1, the same holds for $A^{(r-1)}$. By mathematical induction $A^{(r)}$ is uniquely defined for all $r \leq 0$.

Thus $(r, n) \mapsto A^{(r)}(n)$ is a well-defined map $\mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$.

We have defined $A^{(0)}$ as

$$\forall n \in \mathbb{Z}; A^{(0)}(n) = \begin{cases} n+1 & \text{if } n \leq 0 \\ m & \text{if } n = 1 \\ m \heartsuit^{(0)} n & \text{if } n \geq 2. \end{cases}$$

Using lemma 4.2 and 4.3, by mathematical induction

$$\forall r \in \mathbb{Z} \forall n \in \mathbb{Z}; A^{(r)}(n) = \begin{cases} n+1 & \text{if } n \leq 0 \\ m & \text{if } n = 1 \\ m \heartsuit^{(r)} n & \text{if } n \geq 2. \end{cases}$$

Therefore

$$\forall r \in \mathbb{Z} \forall n \in \mathbb{N}_2; m \heartsuit^{(r)} n = A^{(r)}(n).$$

Since $A^{(r)}(n)$ is uniquely defined and $m\heartsuit^{(r)}n \in \mathbb{N}_2$ for all $r, n \in \mathbb{Z}$, $(r, n) \mapsto m\heartsuit^{(r)}n$ works as a map $\mathbb{Z} \times \mathbb{N}_2 \to \mathbb{N}_2$.

The above holds for all $m \in \mathbb{N}_2$. The map $\mathbb{N}_2 \times \mathbb{Z} \times \mathbb{N}_2 \to \mathbb{N}_2; (m, r, n) \mapsto m\heartsuit^{(r)}n$ is well-defined. □

By proposition 5.1, $\heartsuit^{(1)} = +$. For all $m, n \in \mathbb{N}_2$, and $r \in \mathbb{Z}$;

$$m\heartsuit^{(r+1)}n = \underbrace{m\heartsuit^{(r)}m\heartsuit^{(r)} \cdots \heartsuit^{(r)}m}_{n \text{ copies of } m}.$$

That is an integer extension of the hyperoperation sequence we are finding.

There exist infinitely many bi-infinite sequences of binary operations which have the same property. A binary operation $\star : \mathbb{N}_2 \times \mathbb{N}_2 \to \mathbb{N}_2$ such that

$$m \star n = \begin{cases} m+2 & \text{if } n = m \\ n+1 & \text{if } n \geq m+2 \end{cases}$$

and $A : \mathbb{Z} \to \mathbb{Z}$ is bijective where

$$A(n) = \begin{cases} n+1 & \text{if } n \leq 0 \\ m & \text{if } n = 1 \\ m \star n & \text{if } n \geq 2 \end{cases}$$

can be the 0th element of the bi-infinite hyperoperation sequence, e.g.

$$m \star n = \begin{cases} m-n+1 & \text{if } 2 \leq n \leq m-1 \\ m+2 & \text{if } n = m \\ m+1 & \text{if } n = m+1 \\ n+1 & \text{if } n \geq m+2 \end{cases}$$

$$m \text{♪} n = \begin{cases} n+1 & \text{if } 2 \leq n \leq m-2 \\ 2 & \text{if } n = m-1 \\ m+2 & \text{if } n = m \\ m+1 & \text{if } n = m+1 \\ n+1 & \text{if } n \geq m+2 \end{cases}$$

$$m \text{❊} n = \begin{cases} n+1 & \text{if } 2 \leq n \leq m-2 \\ n+2 & \text{if } n = m-1, m \\ 2 & \text{if } n = m+1 \\ n+1 & \text{if } n \geq m+2. \end{cases}$$

6 Appendices

We calculate −1st to −5th elements of the bi-infinite hyperoperation sequence, where the 0th element is the binary operation ♡ in proposition 5.2.

Negamoniteration

$$A(n) = \begin{cases} & & \text{if } m = 2 \text{ and} \\ n+1 & \leq 1 & n \leq 0 \\ n+1 & = 2 & n = 1 \\ n+2 & = 4 & n = 2 \\ n & = 3 & n = 3 \\ n+1 & \geq 5 & n \geq 4 \\ & & \text{if } m \geq 3 \text{ and} \\ n+1 & \leq 1 & n \leq 0 \\ m & = m & n = 1 \\ n & \geq 2, \leq m-1 & 2 \leq n \leq m-1 \\ n+2 & = m+2 & n = m \\ n & = m+1 & n = m+1 \\ n+1 & \geq m+3 & n \geq m+2 \end{cases}$$

$$A^{-1}(n) = \begin{cases} & & \text{if } m = 2 \text{ and} \\ n-1 & \leq 0 & n \leq 1 \\ n-1 & = 1 & n = 2 \\ n & = 3 & n = 3 \\ n-2 & = 2 & n = 4 \\ n-1 & \geq 4 & n \geq 5 \\ & & \text{if } m \geq 3 \text{ and} \\ n-1 & \leq 0 & n \leq 1 \\ n & \geq 2, \leq m-1 & 2 \leq n \leq m-1 \\ 1 & = 1 & n = m \\ n & = m+1 & n = m+1 \\ n-2 & = m & n = m+2 \\ n-1 & \geq m+2 & n \geq m+3 \end{cases}$$

$$(s \circ A^{-1})(n) = \begin{cases} & & \text{if } m = 2 \text{ and} \\ n & \leq 1 & n \leq 1 \\ n & = 2 & n = 2 \\ n+1 & = 4 & n = 3 \\ n-1 & = 3 & n = 4 \\ n & \geq 5 & n \geq 5 \\ & & \text{if } m \geq 3 \text{ and} \\ n & \leq 1 & n \leq 1 \\ n+1 & \geq 3, \leq m & 2 \leq n \leq m-1 \\ 2 & = 2 & n = m \\ n+1 & = m+2 & n = m+1 \\ n-1 & = m+1 & n = m+2 \\ n & \geq m+3 & n \geq m+3 \end{cases} = \begin{cases} & & \text{if } m = 2 \text{ and} \\ n & \leq 0 & n \leq 0 \\ n & = 1 & n = 1 \\ n & = 2 & n = 2 \\ n+1 & = 4 & n = 3 \\ n-1 & = 3 & n = 4 \\ n & \geq 5 & n \geq 5 \\ & & \text{if } m = 3 \text{ and} \\ n & \leq 0 & n \leq 0 \\ n & = 1 & n = 1 \\ n+1 & = 3 & n = 2 \\ n-1 & = 2 & n = 3 \\ n+1 & = 5 & n = 4 \\ n-1 & = 4 & n = 5 \\ n & \geq 6 & n \geq 6 \\ & & \text{if } m \geq 4 \text{ and} \\ n & \leq 0 & n \leq 0 \\ n & = 1 & n = 1 \\ n+1 & \geq 3, \leq m-1 & 2 \leq n \leq m-2 \\ n+1 & = m & n = m-1 \\ 2 & = 2 & n = m \\ n+1 & = m+2 & n = m+1 \\ n-1 & = m+1 & n = m+2 \\ n & \geq m+3 & n \geq m+3 \end{cases}$$

$$
m\heartsuit^{(-1)}n = A^{(-1)}(n) = (A \circ s \circ A^{-1})(n) = \begin{cases}
 & & \text{if } m = 2 \text{ and} \\
n+1 & \leq 1 & n \leq 0 \\
n+1 & = 2 & n = 1 \\
n+2 & = 4 & n = 2 \\
n+2 & = 5 & n = 3 \\
n-1 & = 3 & n = 4 \\
n+1 & \geq 6 & n \geq 5 \\
 & & \text{if } m = 3 \text{ and} \\
n+1 & \leq 1 & n \leq 0 \\
n+2 & = 3 & n = 1 \\
n+3 & = 5 & n = 2 \\
n-1 & = 2 & n = 3 \\
n+2 & = 6 & n = 4 \\
n-1 & = 4 & n = 5 \\
n+1 & \geq 7 & n \geq 6 \\
 & & \text{if } m \geq 4 \text{ and} \\
n+1 & \leq 1 & n \leq 0 \\
m & = m & n = 1 \\
n+1 & \geq 3, \leq m-1 & 2 \leq n \leq m-2 \\
n+3 & = m+2 & n = m-1 \\
2 & = 2 & n = m \\
n+2 & = m+3 & n = m+1 \\
n-1 & = m+1 & n = m+2 \\
n+1 & \geq m+4 & n \geq m+3
\end{cases}
$$

Negaditeration

$$A^{(-1)-1}(n) = \begin{cases} & & \text{if } m=2 \text{ and} \\ n-1 & \le 0 & n \le 1 \\ n-1 & =1 & n=2 \\ n+1 & =4 & n=3 \\ n-2 & =2 & n=4 \\ n-2 & =3 & n=5 \\ n-1 & \ge 5 & n \ge 6 \\ & & \text{if } m=3 \text{ and} \\ n-1 & \le 0 & n \le 1 \\ n+1 & =3 & n=2 \\ n-2 & =1 & n=3 \\ n+1 & =5 & n=4 \\ n-3 & =2 & n=5 \\ n-2 & =4 & n=6 \\ n-1 & \ge 6 & n \ge 7 \\ & & \text{if } m \ge 4 \text{ and} \\ n-1 & \le 0 & n \le 1 \\ m & =m & n=2 \\ n-1 & \ge 2, \le m-2 & 3 \le n \le m-1 \\ 1 & =1 & n=m \\ n+1 & =m+2 & n=m+1 \\ n-3 & =m-1 & n=m+2 \\ n-2 & =m+1 & n=m+3 \\ n-1 & \ge m+3 & n \ge m+4 \end{cases}$$

$$
(s\circ A^{(-1)-1})(n) = \left\{ \begin{array}{lll}
 & & \text{if } m=2 \text{ and} \\
n & \leq 1 & n \leq 1 \\
n & = 2 & n = 2 \\
n+2 & = 5 & n = 3 \\
n-1 & = 3 & n = 4 \\
n-1 & = 4 & n = 5 \\
n & \geq 6 & n \geq 6 \\
 & & \text{if } m=3 \text{ and} \\
n & \leq 1 & n \leq 1 \\
n+2 & = 4 & n = 2 \\
n-1 & = 2 & n = 3 \\
n+2 & = 6 & n = 4 \\
n-2 & = 3 & n = 5 \\
n-1 & = 5 & n = 6 \\
n & \geq 7 & n \geq 7 \\
 & & \text{if } m \geq 4 \text{ and} \\
n & \leq 1 & n \leq 1 \\
m+1 & = m+1 & n = 2 \\
n & \geq 3, \leq m-1 & 3 \leq n \leq m-1 \\
2 & = 2 & n = m \\
n+2 & = m+3 & n = m+1 \\
n-2 & = m & n = m+2 \\
n-1 & = m+2 & n = m+3 \\
n & \geq m+4 & n \geq m+4
\end{array} \right.
= \left\{ \begin{array}{lll}
 & & \text{if } m=2 \text{ and} \\
n & \leq 0 & n \leq 0 \\
n & = 1 & n = 1 \\
n & = 2 & n = 2 \\
n+2 & = 5 & n = 3 \\
n-1 & = 3 & n = 4 \\
n-1 & = 4 & n = 5 \\
n & \geq 6 & n \geq 6 \\
 & & \text{if } m=3 \text{ and} \\
n & \leq 0 & n \leq 0 \\
n & = 1 & n = 1 \\
n+2 & = 4 & n = 2 \\
n-1 & = 2 & n = 3 \\
n+2 & = 6 & n = 4 \\
n-2 & = 3 & n = 5 \\
n-1 & = 5 & n = 6 \\
n & \geq 7 & n \geq 7 \\
 & & \text{if } m=4 \text{ and} \\
n & \leq 0 & n \leq 0 \\
n & = 1 & n = 1 \\
n+3 & = 5 & n = 2 \\
n & = 3 & n = 3 \\
n-2 & = 2 & n = 4 \\
n+2 & = 7 & n = 5 \\
n-2 & = 4 & n = 6 \\
n-1 & = 6 & n = 7 \\
n & \geq 8 & n \geq 8 \\
 & & \text{if } m \geq 5 \text{ and} \\
n & \leq 0 & n \leq 0 \\
n & = 1 & n = 1 \\
m+1 & = m+1 & n = 2 \\
n & \geq 3, \leq m-2 & 3 \leq n \leq m-2 \\
n & = m-1 & n = m-1 \\
2 & = 2 & n = m \\
n+2 & = m+3 & n = m+1 \\
n-2 & = m & n = m+2 \\
n-1 & = m+2 & n = m+3 \\
n & \geq m+4 & n \geq m+4
\end{array} \right.
$$

$$m\heartsuit^{(-2)}n = A^{(-2)}(n) = (A^{(-1)} \circ s \circ A^{(-1)-1})(n) = \begin{cases} & & \text{if } m = 2 \text{ and} \\ n+1 & \leq 1 & n \leq 0 \\ n+1 & = 2 & n = 1 \\ n+2 & = 4 & n = 2 \\ n+3 & = 6 & n = 3 \\ n+1 & = 5 & n = 4 \\ n-2 & = 3 & n = 5 \\ n+1 & \geq 7 & n \geq 6 \\ & & \text{if } m = 3 \text{ and} \\ n+1 & \leq 1 & n \leq 0 \\ n+2 & = 3 & n = 1 \\ n+4 & = 6 & n = 2 \\ n+2 & = 5 & n = 3 \\ n+3 & = 7 & n = 4 \\ n-3 & = 2 & n = 5 \\ n-2 & = 4 & n = 6 \\ n+1 & \geq 8 & n \geq 7 \\ & & \text{if } m = 4 \text{ and} \\ n+1 & \leq 1 & n \leq 0 \\ n+3 & = 4 & n = 1 \\ n+5 & = 7 & n = 2 \\ n+3 & = 6 & n = 3 \\ n-1 & = 3 & n = 4 \\ n+3 & = 8 & n = 5 \\ n-4 & = 2 & n = 6 \\ n-2 & = 5 & n = 7 \\ n+1 & \geq 9 & n \geq 8 \\ & & \text{if } m \geq 5 \text{ and} \\ n+1 & \leq 1 & n \leq 0 \\ m & = m & n = 1 \\ m+3 & = m+3 & n = 2 \\ n+1 & \geq 4, \leq m-1 & 3 \leq n \leq m-2 \\ n+3 & = m+2 & n = m-1 \\ 3 & = 3 & n = m \\ n+3 & = m+4 & n = m+1 \\ 2 & = 2 & n = m+2 \\ n-2 & = m+1 & n = m+3 \\ n+1 & \geq m+5 & n \geq m+4 \end{cases}$$

Negatriteration

$$A^{(-2)-1}(n) = \begin{cases} & & \text{if } m = 2 \text{ and} \\ n-1 & \leq 0 & n \leq 1 \\ n-1 & = 1 & n = 2 \\ n+2 & = 5 & n = 3 \\ n-2 & = 2 & n = 4 \\ n-1 & = 4 & n = 5 \\ n-3 & = 3 & n = 6 \\ n-1 & \geq 6 & n \geq 7 \\ & & \text{if } m = 3 \text{ and} \\ n-1 & \leq 0 & n \leq 1 \\ n+3 & = 5 & n = 2 \\ n-2 & = 1 & n = 3 \\ n+2 & = 6 & n = 4 \\ n-2 & = 3 & n = 5 \\ n-4 & = 2 & n = 6 \\ n-3 & = 4 & n = 7 \\ n-1 & \geq 7 & n \geq 8 \\ & & \text{if } m = 4 \text{ and} \\ n-1 & \leq 0 & n \leq 1 \\ n+4 & = 6 & n = 2 \\ n+1 & = 4 & n = 3 \\ n-3 & = 1 & n = 4 \\ n+2 & = 7 & n = 5 \\ n-3 & = 3 & n = 6 \\ n-5 & = 2 & n = 7 \\ n-3 & = 5 & n = 8 \\ n-1 & \geq 8 & n \geq 9 \\ & & \text{if } m \geq 5 \text{ and} \\ n-1 & \leq 0 & n \leq 1 \\ m+2 & = m+2 & n = 2 \\ m & = m & n = 3 \\ n-1 & \geq 3, \leq m-2 & 4 \leq n \leq m-1 \\ 1 & = 1 & n = m \\ n+2 & = m+3 & n = m+1 \\ n-3 & = m-1 & n = m+2 \\ 2 & = 2 & n = m+3 \\ n-3 & = m+1 & n = m+4 \\ n-1 & \geq m+4 & n \geq m+5 \end{cases}$$

$$
(s\circ A^{(-2)-1})(n) = \left\{ \begin{array}{lll}
 & & \text{if } m = 2 \text{ and} \\
n & \leq 1 & n \leq 1 \\
n & = 2 & n = 2 \\
n+3 & = 6 & n = 3 \\
n-1 & = 3 & n = 4 \\
n & = 5 & n = 5 \\
n-2 & = 4 & n = 6 \\
n & \geq 7 & n \geq 7 \\
 & & \text{if } m = 3 \text{ and} \\
n & \leq 1 & n \leq 1 \\
n+4 & = 6 & n = 2 \\
n-1 & = 2 & n = 3 \\
n+3 & = 7 & n = 4 \\
n-1 & = 4 & n = 5 \\
n-3 & = 3 & n = 6 \\
n-2 & = 5 & n = 7 \\
n & \geq 8 & n \geq 8 \\
 & & \text{if } m = 4 \text{ and} \\
n & \leq 1 & n \leq 1 \\
n+5 & = 7 & n = 2 \\
n+2 & = 5 & n = 3 \\
n-2 & = 2 & n = 4 \\
n+3 & = 8 & n = 5 \\
n-2 & = 4 & n = 6 \\
n-4 & = 3 & n = 7 \\
n-2 & = 6 & n = 8 \\
n & \geq 9 & n \geq 9 \\
 & & \text{if } m \geq 5 \text{ and} \\
n & \leq 1 & n \leq 1 \\
m+3 & = m+3 & n = 2 \\
m+1 & = m+1 & n = 3 \\
n & \geq 4, \leq m-1 & 4 \leq n \leq m-1 \\
2 & = 2 & n = m \\
n+3 & = m+4 & n = m+1 \\
n-2 & = m & n = m+2 \\
3 & = 3 & n = m+3 \\
n-2 & = m+2 & n = m+4 \\
n & \geq m+5 & n \geq m+5
\end{array} \right.
= \left\{ \begin{array}{lll}
 & & \text{if } m = 2 \text{ and} \\
n & \leq 0 & n \leq 0 \\
n & = 1 & n = 1 \\
n & = 2 & n = 2 \\
n+3 & = 6 & n = 3 \\
n-1 & = 3 & n = 4 \\
n & = 5 & n = 5 \\
n-2 & = 4 & n = 6 \\
n & \geq 7 & n \geq 7 \\
 & & \text{if } m = 3 \text{ and} \\
n & \leq 0 & n \leq 0 \\
n & = 1 & n = 1 \\
n+4 & = 6 & n = 2 \\
n-1 & = 2 & n = 3 \\
n+3 & = 7 & n = 4 \\
n-1 & = 4 & n = 5 \\
n-3 & = 3 & n = 6 \\
n-2 & = 5 & n = 7 \\
n & \geq 8 & n \geq 8 \\
 & & \text{if } m = 4 \text{ and} \\
n & \leq 0 & n \leq 0 \\
n & = 1 & n = 1 \\
n+5 & = 7 & n = 2 \\
n+2 & = 5 & n = 3 \\
n-2 & = 2 & n = 4 \\
n+3 & = 8 & n = 5 \\
n-2 & = 4 & n = 6 \\
n-4 & = 3 & n = 7 \\
n-2 & = 6 & n = 8 \\
n & \geq 9 & n \geq 9 \\
 & & \text{if } m = 5 \text{ and} \\
n & \leq 0 & n \leq 0 \\
n & = 1 & n = 1 \\
n+6 & = 8 & n = 2 \\
n+3 & = 6 & n = 3 \\
n & = 4 & n = 4 \\
n-3 & = 2 & n = 5 \\
n+3 & = 9 & n = 6 \\
n-2 & = 5 & n = 7 \\
n-5 & = 3 & n = 8 \\
n-2 & = 7 & n = 9 \\
n & \geq 10 & n \geq 10 \\
 & & \text{if } m \geq 6 \text{ and} \\
n & \leq 0 & n \leq 0 \\
n & = 1 & n = 1 \\
m+3 & = m+3 & n = 2 \\
m+1 & = m+1 & n = 3 \\
n & \geq 4, \leq m-2 & 4 \leq n \leq m-2 \\
n & = m-1 & n = m-1 \\
2 & = 2 & n = m \\
n+3 & = m+4 & n = m+1 \\
n-2 & = m & n = m+2 \\
3 & = 3 & n = m+3 \\
n-2 & = m+2 & n = m+4 \\
n & \geq m+5 & n \geq m+5
\end{array} \right.
$$

$$m\heartsuit^{(-3)}n = A^{(-3)}(n) = (A^{(-2)}\circ s\circ A^{(-2)-1})(n) = \left\{\begin{array}{lll}
 & & \text{if } m=2 \text{ and} \\
n+1 & \leq 1 & n \leq 0 \\
n+1 & = 2 & n = 1 \\
n+2 & = 4 & n = 2 \\
n+4 & = 7 & n = 3 \\
n+2 & = 6 & n = 4 \\
n-2 & = 3 & n = 5 \\
n-1 & = 5 & n = 6 \\
n+1 & \geq 8 & n \geq 7 \\
 & & \text{if } m=3 \text{ and} \\
n+1 & \leq 1 & n \leq 0 \\
n+2 & = 3 & n = 1 \\
n+2 & = 4 & n = 2 \\
n+3 & = 6 & n = 3 \\
n+4 & = 8 & n = 4 \\
n+2 & = 7 & n = 5 \\
n-1 & = 5 & n = 6 \\
n-5 & = 2 & n = 7 \\
n+1 & \geq 9 & n \geq 8 \\
 & & \text{if } m=4 \text{ and} \\
n+1 & \leq 1 & n \leq 0 \\
n+3 & = 4 & n = 1 \\
n+3 & = 5 & n = 2 \\
n+5 & = 8 & n = 3 \\
n+3 & = 7 & n = 4 \\
n+4 & = 9 & n = 5 \\
n-3 & = 3 & n = 6 \\
n-1 & = 6 & n = 7 \\
n-6 & = 2 & n = 8 \\
n+1 & \geq 10 & n \geq 9 \\
 & & \text{if } m=5 \text{ and} \\
n+1 & \leq 1 & n \leq 0 \\
n+4 & = 5 & n = 1 \\
n+4 & = 6 & n = 2 \\
n+6 & = 9 & n = 3 \\
n+3 & = 7 & n = 4 \\
n+3 & = 8 & n = 5 \\
n+4 & = 10 & n = 6 \\
n-4 & = 3 & n = 7 \\
n-4 & = 4 & n = 8 \\
n-7 & = 2 & n = 9 \\
n+1 & \geq 11 & n \geq 10 \\
 & & \text{if } m \geq 6 \text{ and} \\
n+1 & \leq 1 & n \leq 0 \\
m & = m & n = 1 \\
m+1 & = m+1 & n = 2 \\
m+4 & = m+4 & n = 3 \\
n+1 & \geq 5, \leq m-1 & 4 \leq n \leq m-2 \\
n+3 & = m+2 & n = m-1 \\
n+3 & = m+3 & n = m \\
n+4 & = m+5 & n = m+1 \\
3 & = 3 & n = m+2 \\
4 & = 4 & n = m+3 \\
2 & = 2 & n = m+4 \\
n+1 & \geq m+6 & n \geq m+5
\end{array}\right.$$

Negatetration

$$A^{(-3)-1}(n) = \begin{cases}
 & & \text{if } m = 2 \text{ and} \\
n-1 & \leq 0 & n \leq 1 \\
n-1 & = 1 & n = 2 \\
n+2 & = 5 & n = 3 \\
n-2 & = 2 & n = 4 \\
n+1 & = 6 & n = 5 \\
n-2 & = 4 & n = 6 \\
n-4 & = 3 & n = 7 \\
n-1 & \geq 7 & n \geq 8 \\
 & & \text{if } m = 3 \text{ and} \\
n-1 & \leq 0 & n \leq 1 \\
n+5 & = 7 & n = 2 \\
n-2 & = 1 & n = 3 \\
n-2 & = 2 & n = 4 \\
n+1 & = 6 & n = 5 \\
n-3 & = 3 & n = 6 \\
n-2 & = 5 & n = 7 \\
n-4 & = 4 & n = 8 \\
n-1 & \geq 8 & n \geq 9 \\
 & & \text{if } m = 4 \text{ and} \\
n-1 & \leq 0 & n \leq 1 \\
n+6 & = 8 & n = 2 \\
n+3 & = 6 & n = 3 \\
n-3 & = 1 & n = 4 \\
n-3 & = 2 & n = 5 \\
n+1 & = 7 & n = 6 \\
n-3 & = 4 & n = 7 \\
n-5 & = 3 & n = 8 \\
n-4 & = 5 & n = 9 \\
n-1 & \geq 9 & n \geq 10 \\
 & & \text{if } m = 5 \text{ and} \\
n-1 & \leq 0 & n \leq 1 \\
n+7 & = 9 & n = 2 \\
n+4 & = 7 & n = 3 \\
n+4 & = 8 & n = 4 \\
n-4 & = 1 & n = 5 \\
n-4 & = 2 & n = 6 \\
n-3 & = 4 & n = 7 \\
n-3 & = 5 & n = 8 \\
n-6 & = 3 & n = 9 \\
n-4 & = 6 & n = 10 \\
n-1 & \geq 10 & n \geq 11 \\
 & & \text{if } m \geq 6 \text{ and} \\
n-1 & \leq 0 & n \leq 1 \\
m+4 & = m+4 & n = 2 \\
m+2 & = m+2 & n = 3 \\
m+3 & = m+3 & n = 4 \\
n-1 & \geq 4, \leq m-2 & 5 \leq n \leq m-1 \\
1 & = 1 & n = m \\
2 & = 2 & n = m+1 \\
n-3 & = m-1 & n = m+2 \\
n-3 & = m & n = m+3 \\
3 & = 3 & n = m+4 \\
n-4 & = m+1 & n = m+5 \\
n-1 & \geq m+5 & n \geq m+6
\end{cases}$$

$$
(s\circ A^{(-3)-1})(n) = \left\{ \begin{array}{lll}
 & & \text{if } m=2 \text{ and} \\
n & \le 1 & n \le 1 \\
n & = 2 & n = 2 \\
n+3 & = 6 & n = 3 \\
n-1 & = 3 & n = 4 \\
n+2 & = 7 & n = 5 \\
n-1 & = 5 & n = 6 \\
n-3 & = 4 & n = 7 \\
n & \ge 8 & n \ge 8 \\
 & & \text{if } m=3 \text{ and} \\
n & \le 1 & n \le 1 \\
n+6 & = 8 & n = 2 \\
n-1 & = 2 & n = 3 \\
n-1 & = 3 & n = 4 \\
n+2 & = 7 & n = 5 \\
n-2 & = 4 & n = 6 \\
n-1 & = 6 & n = 7 \\
n-3 & = 5 & n = 8 \\
n & \ge 9 & n \ge 9 \\
 & & \text{if } m=4 \text{ and} \\
n & \le 1 & n \le 1 \\
n+7 & = 9 & n = 2 \\
n+4 & = 7 & n = 3 \\
n-2 & = 2 & n = 4 \\
n-2 & = 3 & n = 5 \\
n+2 & = 8 & n = 6 \\
n-2 & = 5 & n = 7 \\
n-4 & = 4 & n = 8 \\
n-3 & = 6 & n = 9 \\
n & \ge 10 & n \ge 10 \\
 & & \text{if } m=5 \text{ and} \\
n & \le 1 & n \le 1 \\
n+8 & = 10 & n = 2 \\
n+5 & = 8 & n = 3 \\
n+5 & = 9 & n = 4 \\
n-3 & = 2 & n = 5 \\
n-3 & = 3 & n = 6 \\
n-2 & = 5 & n = 7 \\
n-2 & = 6 & n = 8 \\
n-5 & = 4 & n = 9 \\
n-3 & = 7 & n = 10 \\
n & \ge 11 & n \ge 11 \\
 & & \text{if } m \ge 6 \text{ and} \\
n & \le 1 & n \le 1 \\
m+5 & = m+5 & n = 2 \\
m+3 & = m+3 & n = 3 \\
m+4 & = m+4 & n = 4 \\
n & \ge 5, \le m-1 & 5 \le n \le m-1 \\
2 & = 2 & n = m \\
3 & = 3 & n = m+1 \\
n-2 & = m & n = m+2 \\
n-2 & = m+1 & n = m+3 \\
4 & = 4 & n = m+4 \\
n-3 & = m+2 & n = m+5 \\
n & \ge m+6 & n \ge m+6
\end{array} \right.
= \left\{ \begin{array}{lll}
 & & \text{if } m=2 \text{ and} \\
n & \le 0 & n \le 0 \\
n & = 1 & n = 1 \\
n & = 2 & n = 2 \\
n+3 & = 6 & n = 3 \\
n-1 & = 3 & n = 4 \\
n+2 & = 7 & n = 5 \\
n-1 & = 5 & n = 6 \\
n-3 & = 4 & n = 7 \\
n & \ge 8 & n \ge 8 \\
 & & \text{if } m=3 \text{ and} \\
n & \le 0 & n \le 0 \\
n & = 1 & n = 1 \\
n+6 & = 8 & n = 2 \\
n-1 & = 2 & n = 3 \\
n-1 & = 3 & n = 4 \\
n+2 & = 7 & n = 5 \\
n-2 & = 4 & n = 6 \\
n-1 & = 6 & n = 7 \\
n-3 & = 5 & n = 8 \\
n & \ge 9 & n \ge 9 \\
 & & \text{if } m=4 \text{ and} \\
n & \le 0 & n \le 0 \\
n & = 1 & n = 1 \\
n+7 & = 9 & n = 2 \\
n+4 & = 7 & n = 3 \\
n-2 & = 2 & n = 4 \\
n-2 & = 3 & n = 5 \\
n+2 & = 8 & n = 6 \\
n-2 & = 5 & n = 7 \\
n-4 & = 4 & n = 8 \\
n-3 & = 6 & n = 9 \\
n & \ge 10 & n \ge 10 \\
 & & \text{if } m=5 \text{ and} \\
n & \le 0 & n \le 0 \\
n & = 1 & n = 1 \\
n+8 & = 10 & n = 2 \\
n+5 & = 8 & n = 3 \\
n+5 & = 9 & n = 4 \\
n-3 & = 2 & n = 5 \\
n-3 & = 3 & n = 6 \\
n-2 & = 5 & n = 7 \\
n-2 & = 6 & n = 8 \\
n-5 & = 4 & n = 9 \\
n-3 & = 7 & n = 10 \\
n & \ge 11 & n \ge 11 \\
 & & \text{if } m=6 \text{ and} \\
n & \le 0 & n \le 0 \\
n & = 1 & n = 1 \\
n+9 & = 11 & n = 2 \\
n+6 & = 9 & n = 3 \\
n+6 & = 10 & n = 4 \\
n & = 5 & n = 5 \\
n-4 & = 2 & n = 6 \\
n-4 & = 3 & n = 7 \\
n-2 & = 6 & n = 8 \\
n-2 & = 7 & n = 9 \\
n-6 & = 4 & n = 10 \\
n-3 & = 8 & n = 11 \\
n & \ge 12 & n \ge 12 \\
 & & \text{if } m \ge 7 \text{ and} \\
n & \le 0 & n \le 0 \\
n & = 1 & n = 1 \\
m+5 & = m+5 & n = 2 \\
m+3 & = m+3 & n = 3 \\
m+4 & = m+4 & n = 4 \\
n & \ge 5, \le m-2 & 5 \le n \le m-2 \\
n & = m-1 & n = m-1 \\
2 & = 2 & n = m \\
3 & = 3 & n = m+1 \\
n-2 & = m & n = m+2 \\
n-2 & = m+1 & n = m+3 \\
4 & = 4 & n = m+4 \\
n-3 & = m+2 & n = m+5 \\
n & \ge m+6 & n \ge m+6
\end{array} \right.
$$

$$m\heartsuit^{(-4)}n = A^{(-4)}(n) = (A^{(-3)} \circ s \circ A^{(-3)-1})(n) = \left\{\begin{array}{lll}
 & & \text{if } m=2 \text{ and} \\
n+1 & \leq 1 & n \leq 0 \\
n+1 & = 2 & n = 1 \\
n+2 & = 4 & n = 2 \\
n+2 & = 5 & n = 3 \\
n+3 & = 7 & n = 4 \\
n+3 & = 8 & n = 5 \\
n-3 & = 3 & n = 6 \\
n-1 & = 6 & n = 7 \\
n+1 & \geq 9 & n \geq 8 \\
 & & \text{if } m=3 \text{ and} \\
n+1 & \leq 1 & n \leq 0 \\
n+2 & = 3 & n = 1 \\
n+7 & = 9 & n = 2 \\
n+1 & = 4 & n = 3 \\
n+2 & = 6 & n = 4 \\
n-3 & = 2 & n = 5 \\
n+2 & = 8 & n = 6 \\
n-2 & = 5 & n = 7 \\
n-1 & = 7 & n = 8 \\
n+1 & \geq 10 & n \geq 9 \\
 & & \text{if } m=4 \text{ and} \\
n+1 & \leq 1 & n \leq 0 \\
n+3 & = 4 & n = 1 \\
n+8 & = 10 & n = 2 \\
n+3 & = 6 & n = 3 \\
n+1 & = 5 & n = 4 \\
n+3 & = 8 & n = 5 \\
n-4 & = 2 & n = 6 \\
n+2 & = 9 & n = 7 \\
n-1 & = 7 & n = 8 \\
n-6 & = 3 & n = 9 \\
n+1 & \geq 11 & n \geq 10 \\
 & & \text{if } m=5 \text{ and} \\
n+1 & \leq 1 & n \leq 0 \\
n+4 & = 5 & n = 1 \\
n+9 & = 11 & n = 2 \\
n+1 & = 4 & n = 3 \\
n-2 & = 2 & n = 4 \\
n+1 & = 6 & n = 5 \\
n+3 & = 9 & n = 6 \\
n+1 & = 8 & n = 7 \\
n+2 & = 10 & n = 8 \\
n-2 & = 7 & n = 9 \\
n-7 & = 3 & n = 10 \\
n+1 & \geq 12 & n \geq 11 \\
 & & \text{if } m=6 \text{ and} \\
n+1 & \leq 1 & n \leq 0 \\
n+5 & = 6 & n = 1 \\
n+10 & = 12 & n = 2 \\
n+1 & = 4 & n = 3 \\
n-2 & = 2 & n = 4 \\
n+3 & = 8 & n = 5 \\
n+1 & = 7 & n = 6 \\
n+3 & = 10 & n = 7 \\
n+1 & = 9 & n = 8 \\
n+2 & = 11 & n = 9 \\
n-5 & = 5 & n = 10 \\
n-8 & = 3 & n = 11 \\
n+1 & \geq 13 & n \geq 12 \\
 & & \text{if } m \geq 7 \text{ and} \\
n+1 & \leq 1 & n \leq 0 \\
m & = m & n = 1 \\
m+6 & = m+6 & n = 2 \\
n+1 & = 4 & n = 3 \\
n-2 & = 2 & n = 4 \\
n+1 & \geq 6, \leq m-1 & 5 \leq n \leq m-2 \\
n+3 & = m+2 & n = m-1 \\
n+1 & = m+1 & n = m \\
n+3 & = m+4 & n = m+1 \\
n+1 & = m+3 & n = m+2 \\
n+2 & = m+5 & n = m+3 \\
5 & = 5 & n = m+4 \\
3 & = 3 & n = m+5 \\
n+1 & \geq m+7 & n \geq m+6
\end{array}\right.$$

Negapentation

$$
A^{(-4)-1}(n) = \left\{ \begin{array}{lll}
 & & \text{if } m = 2 \text{ and} \\
n-1 & \leq 0 & n \leq 1 \\
n-1 & = 1 & n = 2 \\
n+3 & = 6 & n = 3 \\
n-2 & = 2 & n = 4 \\
n-2 & = 3 & n = 5 \\
n+1 & = 7 & n = 6 \\
n-3 & = 4 & n = 7 \\
n-3 & = 5 & n = 8 \\
n-1 & \geq 8 & n \geq 9 \\
 & & \text{if } m = 3 \text{ and} \\
n-1 & \leq 0 & n \leq 1 \\
n+3 & = 5 & n = 2 \\
n-2 & = 1 & n = 3 \\
n-1 & = 3 & n = 4 \\
n+2 & = 7 & n = 5 \\
n-2 & = 4 & n = 6 \\
n+1 & = 8 & n = 7 \\
n-2 & = 6 & n = 8 \\
n-7 & = 2 & n = 9 \\
n-1 & \geq 9 & n \geq 10 \\
 & & \text{if } m = 4 \text{ and} \\
n-1 & \leq 0 & n \leq 1 \\
n+4 & = 6 & n = 2 \\
n+6 & = 9 & n = 3 \\
n-3 & = 1 & n = 4 \\
n-1 & = 4 & n = 5 \\
n-3 & = 3 & n = 6 \\
n+1 & = 8 & n = 7 \\
n-3 & = 5 & n = 8 \\
n-2 & = 7 & n = 9 \\
n-8 & = 2 & n = 10 \\
n-1 & \geq 10 & n \geq 11 \\
 & & \text{if } m = 5 \text{ and} \\
n-1 & \leq 0 & n \leq 1 \\
n+2 & = 4 & n = 2 \\
n+7 & = 10 & n = 3 \\
n-1 & = 3 & n = 4 \\
n-4 & = 1 & n = 5 \\
n-1 & = 5 & n = 6 \\
n+2 & = 9 & n = 7 \\
n-1 & = 7 & n = 8 \\
n-3 & = 6 & n = 9 \\
n-2 & = 8 & n = 10 \\
n-9 & = 2 & n = 11 \\
n-1 & \geq 11 & n \geq 12 \\
 & & \text{if } m = 6 \text{ and} \\
n-1 & \leq 0 & n \leq 1 \\
n+2 & = 4 & n = 2 \\
n+8 & = 11 & n = 3 \\
n-1 & = 3 & n = 4 \\
n+5 & = 10 & n = 5 \\
n-5 & = 1 & n = 6 \\
n-1 & = 6 & n = 7 \\
n-3 & = 5 & n = 8 \\
n-1 & = 8 & n = 9 \\
n-3 & = 7 & n = 10 \\
n-2 & = 9 & n = 11 \\
n-10 & = 2 & n = 12 \\
n-1 & \geq 12 & n \geq 13 \\
 & & \text{if } m \geq 7 \text{ and} \\
n-1 & \leq 0 & n \leq 1 \\
n+2 & = 4 & n = 2 \\
m+5 & = m+5 & n = 3 \\
n-1 & = 3 & n = 4 \\
m+4 & = m+4 & n = 5 \\
n-1 & \geq 5, \leq m-2 & 6 \leq n \leq m-1 \\
1 & = 1 & n = m \\
n-1 & = m & n = m+1 \\
n-3 & = m-1 & n = m+2 \\
n-1 & = m+2 & n = m+3 \\
n-3 & = m+1 & n = m+4 \\
n-2 & = m+3 & n = m+5 \\
2 & = 2 & n = m+6 \\
n-1 & \geq m+6 & n \geq m+7
\end{array} \right.
$$

$$
(s\circ A^{(-4)-1})(n)=\left\{\begin{array}{lll}
 & & \text{if } m=2 \text{ and}\\
n & \leq 1 & n\leq 1\\
n & =2 & n=2\\
n+4 & =7 & n=3\\
n-1 & =3 & n=4\\
n-1 & =4 & n=5\\
n+2 & =8 & n=6\\
n-2 & =5 & n=7\\
n-2 & =6 & n=8\\
n & \geq 9 & n\geq 9\\
 & & \text{if } m=3 \text{ and}\\
n & \leq 1 & n\leq 1\\
n+4 & =6 & n=2\\
n-1 & =2 & n=3\\
n & =4 & n=4\\
n+3 & =8 & n=5\\
n-1 & =5 & n=6\\
n+2 & =9 & n=7\\
n-1 & =7 & n=8\\
n-6 & =3 & n=9\\
n & \geq 10 & n\geq 10\\
 & & \text{if } m=4 \text{ and}\\
n & \leq 1 & n\leq 1\\
n+5 & =7 & n=2\\
n+7 & =10 & n=3\\
n-2 & =2 & n=4\\
n & =5 & n=5\\
n-2 & =4 & n=6\\
n+2 & =9 & n=7\\
n-2 & =6 & n=8\\
n-1 & =8 & n=9\\
n-7 & =3 & n=10\\
n & \geq 11 & n\geq 11\\
 & & \text{if } m=5 \text{ and}\\
n & \leq 1 & n\leq 1\\
n+3 & =5 & n=2\\
n+8 & =11 & n=3\\
n & =4 & n=4\\
n-3 & =2 & n=5\\
n & =6 & n=6\\
n+3 & =10 & n=7\\
n & =8 & n=8\\
n-2 & =7 & n=9\\
n-1 & =9 & n=10\\
n-8 & =3 & n=11\\
n & \geq 12 & n\geq 12\\
 & & \text{if } m=6 \text{ and}\\
n & \leq 1 & n\leq 1\\
n+3 & =5 & n=2\\
n+9 & =12 & n=3\\
n & =4 & n=4\\
n+6 & =11 & n=5\\
n-4 & =2 & n=6\\
n & =7 & n=7\\
n-2 & =6 & n=8\\
n & =9 & n=9\\
n-2 & =8 & n=10\\
n-1 & =10 & n=11\\
n-9 & =3 & n=12\\
n & \geq 13 & n\geq 13\\
 & & \text{if } m\geq 7 \text{ and}\\
n & \leq 1 & n\leq 1\\
n+3 & =5 & n=2\\
m+6 & =m+6 & n=3\\
n & =4 & n=4\\
m+5 & =m+5 & n=5\\
n & \geq 6, \leq m-1 & 6\leq n\leq m-1\\
2 & =2 & n=m\\
n & =m+1 & n=m+1\\
n-2 & =m & n=m+2\\
n & =m+3 & n=m+3\\
n-2 & =m+2 & n=m+4\\
n-1 & =m+4 & n=m+5\\
3 & =3 & n=m+6\\
n & \geq m+7 & n\geq m+7
\end{array}\right.
=\left\{\begin{array}{lll}
 & & \text{if } m=2 \text{ and}\\
n & \leq 0 & n\leq 0\\
n & =1 & n=1\\
n & =2 & n=2\\
n+4 & =7 & n=3\\
n-1 & =3 & n=4\\
n-1 & =4 & n=5\\
n+2 & =8 & n=6\\
n-2 & =5 & n=7\\
n-2 & =6 & n=8\\
n & \geq 9 & n\geq 9\\
 & & \text{if } m=3 \text{ and}\\
n & \leq 0 & n\leq 0\\
n & =1 & n=1\\
n+4 & =6 & n=2\\
n-1 & =2 & n=3\\
n & =4 & n=4\\
n+3 & =8 & n=5\\
n-1 & =5 & n=6\\
n+2 & =9 & n=7\\
n-1 & =7 & n=8\\
n-6 & =3 & n=9\\
n & \geq 10 & n\geq 10\\
 & & \text{if } m=4 \text{ and}\\
n & \leq 0 & n\leq 0\\
n & =1 & n=1\\
n+5 & =7 & n=2\\
n+7 & =10 & n=3\\
n-2 & =2 & n=4\\
n & =5 & n=5\\
n-2 & =4 & n=6\\
n+2 & =9 & n=7\\
n-2 & =6 & n=8\\
n-1 & =8 & n=9\\
n-7 & =3 & n=10\\
n & \geq 11 & n\geq 11\\
 & & \text{if } m=5 \text{ and}\\
n & \leq 0 & n\leq 0\\
n & =1 & n=1\\
n+3 & =5 & n=2\\
n+8 & =11 & n=3\\
n & =4 & n=4\\
n-3 & =2 & n=5\\
n & =6 & n=6\\
n+3 & =10 & n=7\\
n & =8 & n=8\\
n-2 & =7 & n=9\\
n-1 & =9 & n=10\\
n-8 & =3 & n=11\\
n & \geq 12 & n\geq 12\\
 & & \text{if } m=6 \text{ and}\\
n & \leq 0 & n\leq 0\\
n & =1 & n=1\\
n+3 & =5 & n=2\\
n+9 & =12 & n=3\\
n & =4 & n=4\\
n+6 & =11 & n=5\\
n-4 & =2 & n=6\\
n & =7 & n=7\\
n-2 & =6 & n=8\\
n & =9 & n=9\\
n-2 & =8 & n=10\\
n-1 & =10 & n=11\\
n-9 & =3 & n=12\\
n & \geq 13 & n\geq 13\\
 & & \text{if } m=7 \text{ and}\\
n & \leq 0 & n\leq 0\\
n & =1 & n=1\\
n+3 & =5 & n=2\\
n+10 & =13 & n=3\\
n & =4 & n=4\\
n+7 & =12 & n=5\\
n & =6 & n=6\\
n-5 & =2 & n=7\\
n & =8 & n=8\\
n-2 & =7 & n=9\\
n & =10 & n=10\\
n-2 & =9 & n=11\\
n-1 & =11 & n=12\\
n-10 & =3 & n=13\\
n & \geq 14 & n\geq 14\\
 & & \text{if } m\geq 8 \text{ and}\\
n & \leq 0 & n\leq 0\\
n & =1 & n=1\\
n+3 & =5 & n=2\\
m+6 & =m+6 & n=3\\
n & =4 & n=4\\
m+5 & =m+5 & n=5\\
n & \geq 6, \leq m-2 & 6\leq n\leq m-2\\
n & =m-1 & n=m-1\\
2 & =2 & n=m\\
n & =m+1 & n=m+1\\
n-2 & =m & n=m+2\\
n & =m+3 & n=m+3\\
n-2 & =m+2 & n=m+4\\
n-1 & =m+4 & n=m+5\\
3 & =3 & n=m+6\\
n & \geq m+7 & n\geq m+7
\end{array}\right.
$$

$$
m\heartsuit^{(-5)}n = A^{(-5)}(n) = (A^{(-4)} \circ s \circ A^{(-4)-1})(n) = \left\{\begin{array}{lll}
 & & \text{if } m = 2 \text{ and} \\
n+1 & \leq 1 & n \leq 0 \\
n+1 & = 2 & n = 1 \\
n+2 & = 4 & n = 2 \\
n+3 & = 6 & n = 3 \\
n+1 & = 5 & n = 4 \\
n+2 & = 7 & n = 5 \\
n+3 & = 9 & n = 6 \\
n+1 & = 8 & n = 7 \\
n-5 & = 3 & n = 8 \\
n+1 & \geq 10 & n \geq 9 \\
 & & \text{if } m = 3 \text{ and} \\
n+1 & \leq 1 & n \leq 0 \\
n+2 & = 3 & n = 1 \\
n+6 & = 8 & n = 2 \\
n+6 & = 9 & n = 3 \\
n+2 & = 6 & n = 4 \\
n+2 & = 7 & n = 5 \\
n-4 & = 2 & n = 6 \\
n+3 & = 10 & n = 7 \\
n-3 & = 5 & n = 8 \\
n-5 & = 4 & n = 9 \\
n+1 & \geq 11 & n \geq 10 \\
 & & \text{if } m = 4 \text{ and} \\
n+1 & \leq 1 & n \leq 0 \\
n+3 & = 4 & n = 1 \\
n+7 & = 9 & n = 2 \\
n+8 & = 11 & n = 3 \\
n+6 & = 10 & n = 4 \\
n+3 & = 8 & n = 5 \\
n-1 & = 5 & n = 6 \\
n-4 & = 3 & n = 7 \\
n-6 & = 2 & n = 8 \\
n-2 & = 7 & n = 9 \\
n-4 & = 6 & n = 10 \\
n+1 & \geq 12 & n \geq 11 \\
 & & \text{if } m = 5 \text{ and} \\
n+1 & \leq 1 & n \leq 0 \\
n+4 & = 5 & n = 1 \\
n+4 & = 6 & n = 2 \\
n+9 & = 12 & n = 3 \\
n-2 & = 2 & n = 4 \\
n+6 & = 11 & n = 5 \\
n+3 & = 9 & n = 6 \\
n-4 & = 3 & n = 7 \\
n+2 & = 10 & n = 8 \\
n-1 & = 8 & n = 9 \\
n-3 & = 7 & n = 10 \\
n-7 & = 4 & n = 11 \\
n+1 & \geq 13 & n \geq 12 \\
 & & \text{if } m = 6 \text{ and} \\
n+1 & \leq 1 & n \leq 0 \\
n+5 & = 6 & n = 1 \\
n+6 & = 8 & n = 2 \\
n+10 & = 13 & n = 3 \\
n-2 & = 2 & n = 4 \\
n-2 & = 3 & n = 5 \\
n+6 & = 12 & n = 6 \\
n+3 & = 10 & n = 7 \\
n-1 & = 7 & n = 8 \\
n+2 & = 11 & n = 9 \\
n-1 & = 9 & n = 10 \\
n-6 & = 5 & n = 11 \\
n-8 & = 4 & n = 12 \\
n+1 & \geq 14 & n \geq 13 \\
 & & \text{if } m = 7 \text{ and} \\
n+1 & \leq 1 & n \leq 0 \\
n+6 & = 7 & n = 1 \\
n+4 & = 6 & n = 2 \\
n+11 & = 14 & n = 3 \\
n-2 & = 2 & n = 4 \\
n-2 & = 3 & n = 5 \\
n+3 & = 9 & n = 6 \\
n+6 & = 13 & n = 7 \\
n+3 & = 11 & n = 8 \\
n-1 & = 8 & n = 9 \\
n+2 & = 12 & n = 10 \\
n-1 & = 10 & n = 11 \\
n-7 & = 5 & n = 12 \\
n-9 & = 4 & n = 13 \\
n+1 & \geq 15 & n \geq 14 \\
 & & \text{if } m \geq 8 \text{ and} \\
n+1 & \leq 1 & n \leq 0 \\
m & = m & n = 1 \\
n+4 & = 6 & n = 2 \\
m+7 & = m+7 & n = 3 \\
n-2 & = 2 & n = 4 \\
n-2 & = 3 & n = 5 \\
n+1 & \geq 7, \leq m-1 & 6 \leq n \leq m-2 \\
n+3 & = m+2 & n = m-1 \\
n+6 & = m+6 & n = m \\
n+3 & = m+4 & n = m+1 \\
n-1 & = m+1 & n = m+2 \\
n+2 & = m+5 & n = m+3 \\
n-1 & = m+3 & n = m+4 \\
5 & = 5 & n = m+5 \\
4 & = 4 & n = m+6 \\
n+1 & \geq m+8 & n \geq m+7
\end{array}\right.
$$

Postscript

I am translating ” ハイパー演算の整数拡張” which I wrote in japanese last year.

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