# Strange symmetries in different results

## 1 – The Euler's observations

It was indeed been detected by Euler that the real roots of functional equation:

$$y = b^y \qquad (1)$$

don't always include some real solutions of:

$$y = b^{b^y}\ . \qquad (2)$$

This is certainly "dynamically" right, if we think of (1) and (2) as, for instance, instructions of a programming language or as the description of iterative cycles. Nevertheless, "*statically*", this fact is very peculiar, because the two functional equations are identical. In fact, by substitution, we get:

$$y = b^y = b^{b^y} \qquad (3)$$

Actually, with my ... simple mind, I would suggest that we might even assume that:

$$y = b^y = b^{b^y} = b^{b^{b^y}} = \ldots = b^{b^{\cdot^{\cdot^{y}}}} = \lim_{n\to\infty} {}^n b = {}^\infty b \qquad (4)$$

In other words, if: $y = b^y$

then, at the end of the "day": $y = {}^\infty b$ ($y$ is the *infinite tetrate* of $b$)[1]

and, from (1), we get $b = \sqrt[y]{y}$ ($b$ is the *selfroot* of $y$) (5)

but also, from (3) and (5): $b = {}^\infty\!\!\sqrt{y} = \mathrm{srt}_\infty\, y$ ($b$ is the *infinite superroot* of $y$)

With the help of the Lambert Function, we also know that we can obtain the inverse of (5), as follows:

$$y = {}^\infty b = \mathrm{plog}(-\ln b)/(-\ln b) \qquad (6)$$

The analysis of (6) is extremely complicated, because "plog(z)" is multi-valued complex function, with two real branches given by the *Mathematica* operators as:

$$\mathrm{plog}_0(z) = \mathrm{ProductLog}[z]$$
$$\mathrm{plog}_{-1}(z) = \mathrm{ProductLog}[-1, z] \qquad (7)$$

However, infinite other branches are available in *Mathematica* as $\mathrm{ProductLog}[k, z]$, giving the $k$-th order branch of the Lambert Function (to be further analyzed). The two real branches (7) allow us to plot the inverse of (5) as follows:

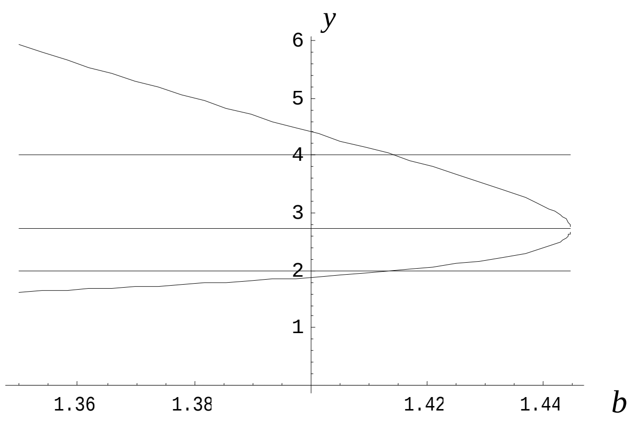


Fig. 1

Fig. 1 shows part of the two (upper and lower) real branches of $y = {}^\infty b$ obtained by (6) in the domain:

$$1 < b < \eta\ ,\ \text{with}\ \eta = \sqrt[e]{e} = 1.444667861.. \qquad (8)$$

The figure also shows the upper and lower values of the infinite *tetrates* (*towers*) of $b = \sqrt{2}$ , {2, 4}.

[1] Please, don't *kill* the word "**tower**", similar, at a lower hyperoperation rank, to "**power**".

## 2 – The “Yellow Zone”

Now, let us indicate the tetration to the base *b* of super-exponent *x*, as follows:

$$\boxed{y = {}^{x}b} \tag{9}$$

We know that the plotting of *tetrates* (*towers*) with positive integer super-exponents *x* = *n* reveal what I called “the yellow zone”, the “of-limits area for ... positive integers”[2], for $n \to +\infty$. Let see the following plots, $y(b)$ and $b(y)$, both in their respective 0...1 domains:

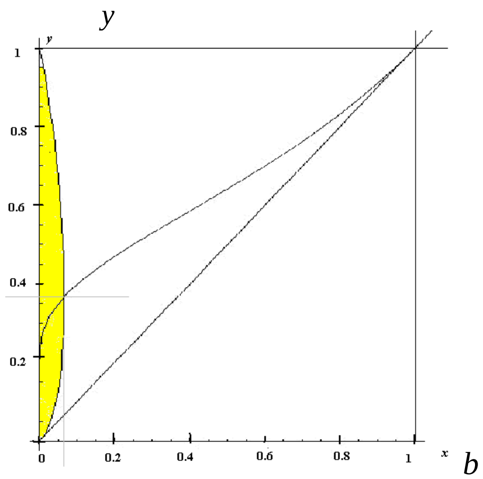


Fig. 2

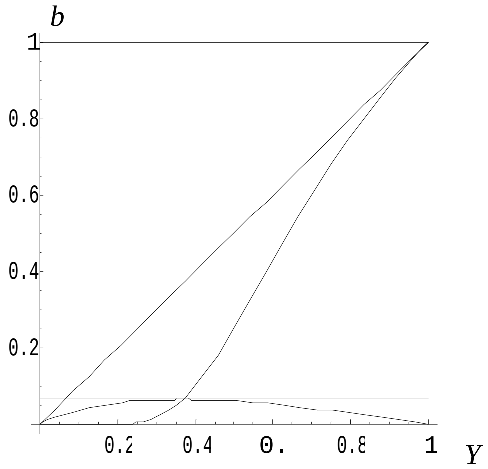


Fig. 3

Fig. 2 plot shows the yellow area in diagram $y(b)$, in the domain:

$$0 < b < \beta \text{, with } \beta = e^{-e} = 0.065988036.. \tag{10}$$

with a maximum for *b* (bifurcation point) in that area at the abscissa:

$$b = \beta = e^{-e} = 0.065988036..$$

and at the ordinate $y = 1/e = 0.367879441..$

Fig. 3 plot shows $b(y)$ with the inverse “yellow area”, with a maximum at:

$$b = 1/e = 0.367879441...$$

at the abscissa: $y = \beta = e^{-e} = 0.065988036..$

With reference to the second plot, we can define a $b(y) = y^{1/y}$ function such as:

$$b(e) = \sqrt[e]{e} = \eta \tag{11}$$

$$b(1/e) = e^{-e} = \beta$$

[2] This is a good one!

## 2 – The Andydude's and GFR's proposals

Andrew's developments showing the famous transition (yellow ...) zone, by solving equation (2):

$$\boxed{x = b^{b^x}} \qquad (12)$$

This solution gives the following plot (© Andydude):

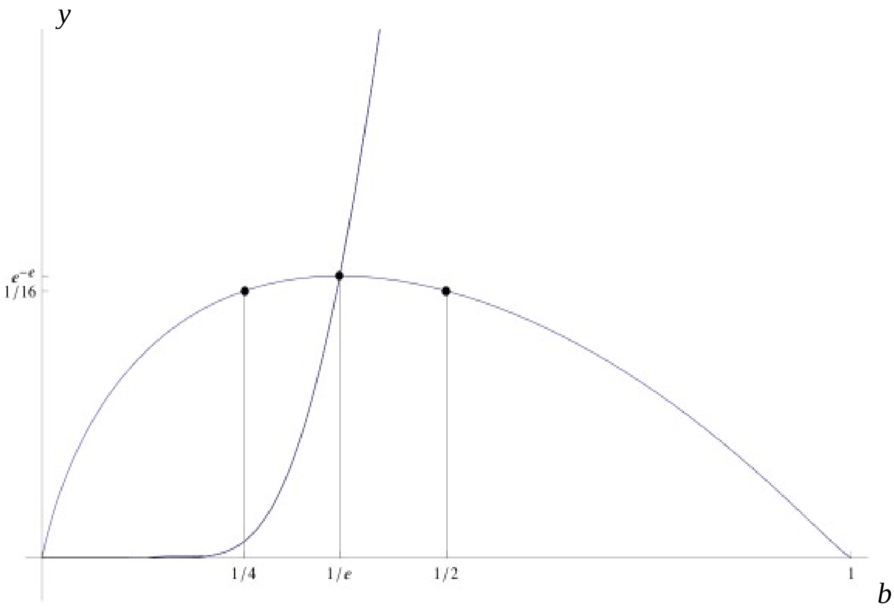


Fig. 4

Andrew's main observation is that the maximum is $y = \eta = 0.065988036.. = 1/\sim 15.1543..$ has the expected coordinates ($b = 1/e$) described by (6) and that two almost symmetrical points are found for $b = 1/4$ and $b = 1/2$, both defining one value of $y = 0.0625 = 1/16$.

GFR plot obtained with formula :

$$\boxed{y = \kappa \cdot (b^{-b} - 1)}, \qquad \text{with } \kappa = \frac{e^{-e}}{e^{1/e} - 1} = \frac{\beta}{\eta - 1} \qquad (13)$$

shows a similar display (by *Mathematica*):

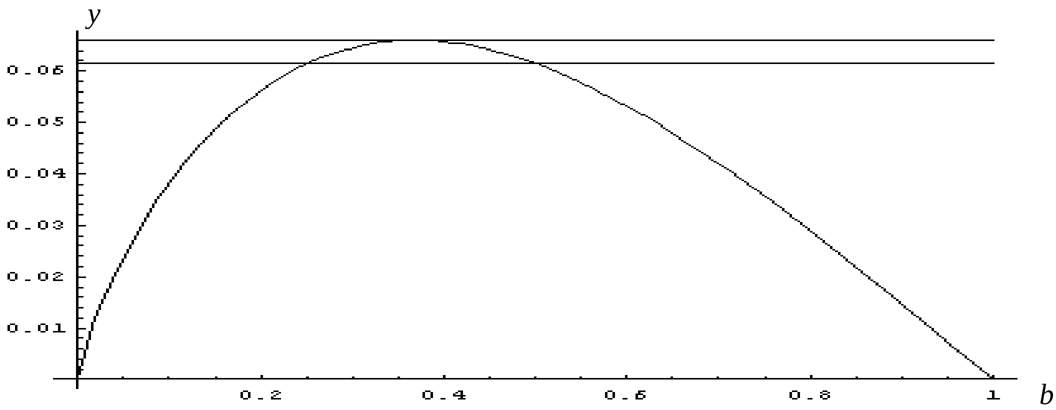


Fig. 5

The coordinates of the maximum are the same and there are also two values of *b*, $b = 1/4$ and $b = 1/2$ that define the same ordinate $y = 0.0614686641382 65.. = 1/\sim 16.268..$, obviously different from the "round" 1/16, as found by Andrew. This magic phenomenon is due to the $b^{-b}$ expression appearing in (13), which gives $\sqrt{2}$ for $b = \{0.25,\ 0.5\}$.

3 – <u>An implementation of $y$ = b-tetra-$x$, for $b$ = 0.06146864138265</u>..

Let us consider again (1), $y = {}^{x}b$, with $b$ = 0.06146864138265.., as defined in the previous section. With reference to Fig. 2, this corresponds to a $yx$ cross section passing through the "yellow zone" at that constant value of $b < \beta$. By supposing to simulate the "critical path" in the domain $-1 < x < 0$ with a sinusoidal wave, we get the following extended plot, for -2 < $x$ < 40, by using $y(x+1) = b^{y(x)}$ and $y(x-1) = \log_b y(x)$:

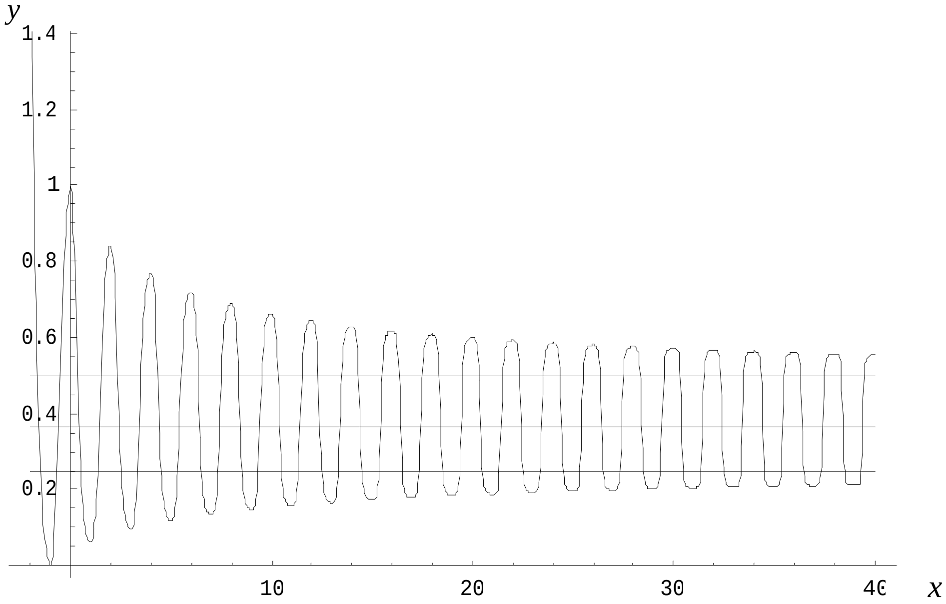

Fig. 6

Fig. 6 plot also shows the asymptotical trend of maximum and minimum elongations of the persistent $y$ oscillations, at $y$ = ¼ and $y$ = ½, for $x \to +\infty$. Fig 7 shows the same plot for -2 < $x$ < 250. The mid-range value is $y$ = 0.363155637096505..., as foreseen.

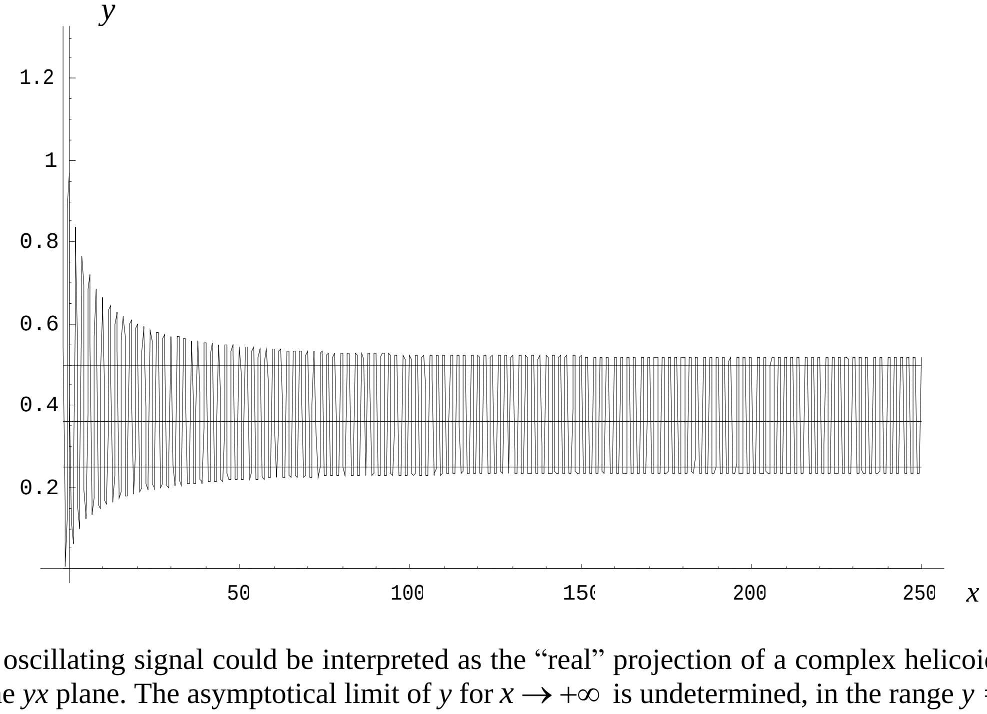

Fig. 7

This oscillating signal could be interpreted as the "real" projection of a complex helicoidal magnitude on the $yx$ plane. The asymptotical limit of $y$ for $x \to +\infty$ is undetermined, in the range $y$ = [0.25-0.50].

4 – An implementation of $y = b$-tetra-$x$, for $\beta < b < 1$.

As a matter of comparison, recalling that $\beta = e^{-e} = 0.065988036..$, it is interesting to apply the same simulation method for base $b = 0.5$ (asymptotic $y =$ 0.641185744504986 ) :

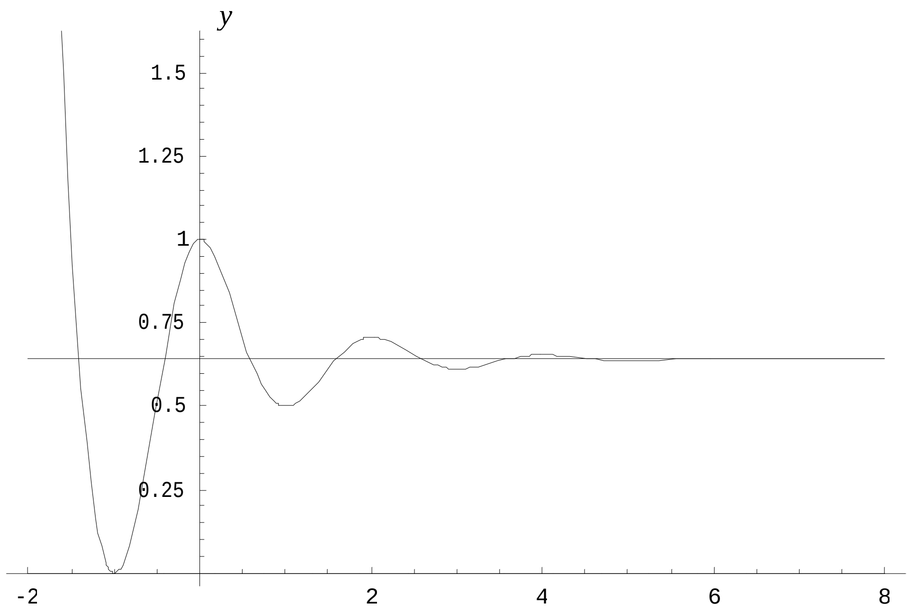


Fig. 8

A similar simulation for $b = 0.8$ gives (asymptotic $y =$ 0.38151533939505816 ):

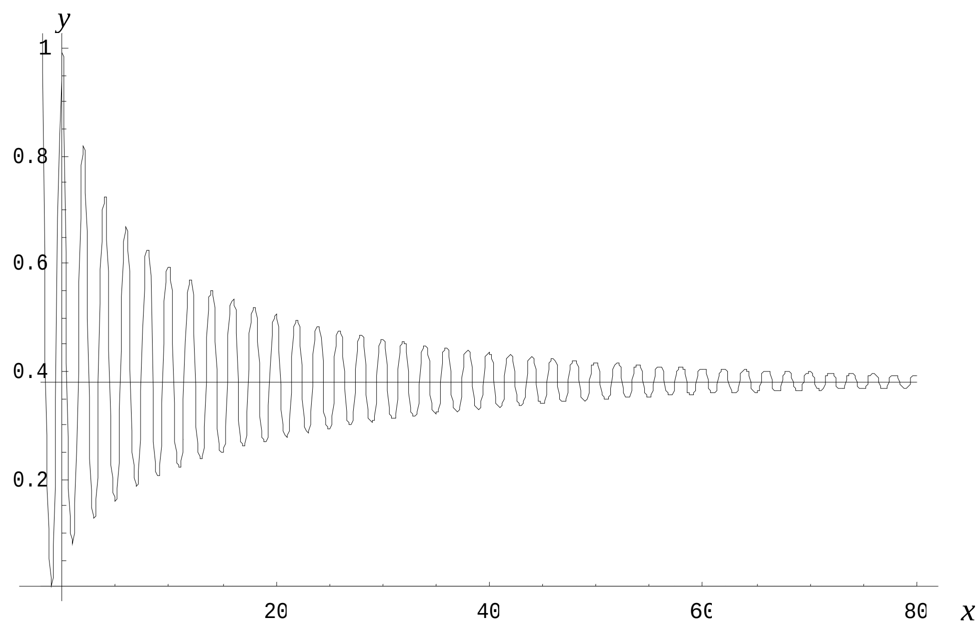


Fig. 9

Both plots are for $\beta < b < 1$. The asymptotic $y$ values are given by $y = {}^{\infty}b = \text{plog}\left(-\ln b\right)/(-\ln b)$.

We observe that, for $0 < b < 1$: $\lim_{x \to -2^+} {}^{x}b = +\infty$ and: for $1 < b < \eta$: $\lim_{x \to -2^+} {}^{x}b = -\infty$. The case of base $b = 1$ is singular. The limits for $x \to -2$ are calculated "from the right". In the domain $x < -2$, the tetrational functions imply the use of the logarithms of negative numbers.

**GFR – 2008-01-20**