## 1. Test of Borel-summation on half-iterate *g(z)* of *f(z) = exp(z)-1* and *g(g(z))=f(z)* at *z=1*

| *k* | "Standard handwave" sum [1] | | *x* | Borelsum [2] |
|---|---|---|---|---|
| 0 | 0 | | 1 | 0.40752231461956581918640076515 7 |
| 1 | 0.476190476190476190476190476190 | | 2 | 0.799759762925598692863721071948 |
| 2 | 0.752362639428100165949346099524 | | 3 | 1.01908443987779069015189021525 |
| 3 | 0.923050527876717167299566027648 | | 4 | 1.13720236485406900457772408949 |
| 4 | 1.03289231796767618222551176392 | | 5 | 1.20084366531830464027742410553 |
| 5 | 1.10558310438551565267455758169 | | 6 | 1.23482260169649441695444633526 |
| 6 | 1.15469209171100397593035484350 | | 7 | 1.25266418145291395604642178030 |
| 7 | 1.18840535763204389314631546587 | | 8 | 1.26185864637806098613795087792 |
| 8 | 1.21185032739741380707011983582 | | 9 | 1.26651312401874568249145429628 |
| 9 | 1.22833039323199882236844617800 | | 10 | 1.26883204988561525614464699375 |
| 10 | 1.24002094539231147610897490782 | | 11 | 1.26997136783657473590675490193 |
| 11 | 1.24837993312456182602551593590 | | 12 | 1.27052435385551596246122324889 |
| 12 | 1.25439869780849919135300652257 | | 13 | 1.27078988644241333811041175218 |
| 13 | 1.25875955055838927759923344144 | | 14 | 1.27091616811556608526441805631 |
| 14 | 1.26193702343835432434988886861 | | 15 | 1.27097570105545584555728520473 |
| ... | ... | | ... | ... |
| 227 | 1.271027413889951521 4 1690087903 | | 58 | 1.271027413889951521 4 1595811241 |
| 228 | 1.2710274138899515214 1803069632 | | 59 | 1.27102741388995152142 106871190 |
| 229 | 1.2710274138899515214 1899480480 | | 60 | 1.27102741388995152142 317669637 |
| 230 | 1.2710274138899515214 1981766088 | | 61 | 1.271027413889951521424 04520391 |
| 231 | 1.27102741388995152142 052008875 | | 62 | 1.271027413889951521424 40264332 |
| 232 | 1.27102741388995152142 111982324 | | 63 | 1.271027413889951521424 54959193 |
| 233 | 1.27102741388995152142 163197109 | | 64 | 1.271027413889951521424 6 0994145 |
| 234 | 1.27102741388995152142 206940285 | | 65 | 1.271027413889951521424 6 3470073 |
| 235 | 1.27102741388995152142 244308597 | | 66 | 1.271027413889951521424 6 4484846 |
| 236 | 1.27102741388995152142 276236799 | | 67 | 1.271027413889951521424 6 4900350 |
| 237 | 1.27102741388995152142 303521730 | | 68 | 1.271027413889951521424 65 070318 |
| 238 | 1.27102741388995152142 326842789 | | 69 | 1.271027413889951521424 651 39781 |
| 239 | 1.27102741388995152142 346779353 | | 70 | 1.271027413889951521424 651 68144 |
| 240 | 1.27102741388995152142 363825592 | | 71 | 1.271027413889951521424 651 79714 |
| 241 | 1.27102741388995152142 378403081 | | 72 | 1.271027413889951521424 6518 4430 |
| 242 | 1.27102741388995152142 390871526 | | 73 | 1.271027413889951521424 6518 6351 |
| 243 | 1.271027413889951521424 01537905 | | 74 | 1.271027413889951521424 65187 132 |
| 244 | 1.271027413889951521424 10664239 | | 75 | 1.271027413889951521424 65187 450 |
| 245 | 1.271027413889951521424 18474224 | | 76 | 1.271027413889951521424 65187 579 |
| 246 | 1.271027413889951521424 25158861 | | 77 | 1.271027413889951521424 651876 32 |
| 247 | 1.271027413889951521424 30881275 | | 78 | 1.271027413889951521424 651876 53 |
| 248 | 1.271027413889951521424 35780798 | | 79 | 1.271027413889951521424 651876 61 |
| 249 | 1.271027413889951521424 39976468 | | 80 | 1.271027413889951521424 6518766 5 |
| 250 | 1.271027413889951521424 43569998 | | 81 | 1.271027413889951521424 6518766 6 |
| 251 | 1.271027413889951521424 46648316 | | 82 | 1.271027413889951521424 **65187667** |
| 252 | 1.271027413889951521424 49285726 | | 83 | 1.271027413889951521424 **65187667** |
| 253 | 1.271027413889951521424 **51545750** | | 84 | 1.271027413889951521424 **65187667** |
| 254 | 1.271027413889951521424 **53482706** | | 85 | 1.271027413889951521424 **65187667** |
| 255 | 1.271027413889951521424 **55143047** | | 86 | 1.271027413889951521424 **65187667** |
| ... | ... | | ... | ... |

**"Standard-Handwave"-sum:**

*experimental Noerlund sum implementation using modified Pascalmatrix*

(1) *partial sums up to index k=255.*
*Used parameters PkPowSum(ord=1.445, eord=1.1, n=256)*

**Borel sum:**

*After description in K. Knopp, chap 13*

(2) *Borel with order r=2, using the partial sums as terms (as given in Knopp-formula) using 512 terms for the sum in numerator as well as in denominator*

(3) *Borel with order r=2, using the partial sums as terms (as given in Knopp-formula) using (a)1024,(b)128 terms for the sum in numerator as well as in denominator*

| x | $\text{Borel}^{(3a)}$ --> (52 digits correct) |
|---|---|
| 129 | 1.2710274138899515214246518766722806879544209972104 |
| 130 | 1.2710274138899515214246518766722806879544209972108 |
| 131 | 1.2710274138899515214246518766722806879544209972109 |
| 132 | 1.2710274138899515214246518766722806879544209972110 |
| 133 | 1.2710274138899515214246518766722806879544209972110 |
| 134 | 1.2710274138899515214246518766722806879544209972110 |
| 135 | 1.2710274138899515214246518766722806879544209972110 |
| 136 | 1.2710274138899515214246518766722806879544209972110 |
| 137 | 1.2710274138899515214246518766722806879544209972110 |
| 138 | 1.2710274138899515214246518766722806879544209972110 |

## Using functional equation to insert $z_{-40}=f^{\circ -40}(1)$ instead of $z=1$

This is allowed ***after*** we have shown, that the Borel-summation really transforms $g(z)$ into a convergent series-expression: that means the growthrate of the coefficients of the formal powerseries of $g(z)$ ***has been shown*** to be not more than $O(c^k\, k!)$ .

| x | $\text{Borel}^{(3b)}$ using $f^{\circ 40}(\, g(z_{-40})\, )$ (gives 70 digits) |
|---|---|
| 169 | 1.2710274138899515214246518766722806879544209972110290192136486991790010 |
| 170 | 1.2710274138899515214246518766722806879544209972110290192136486991790015 |
| 171 | 1.2710274138899515214246518766722806879544209972110290192136486991790017 |
| 172 | 1.2710274138899515214246518766722806879544209972110290192136486991790018 |
| 173 | 1.2710274138899515214246518766722806879544209972110290192136486991790019 |
| 174 | 1.2710274138899515214246518766722806879544209972110290192136486991790019 |
| 175 | 1.2710274138899515214246518766722806879544209972110290192136486991790019 |
| 176 | 1.2710274138899515214246518766722806879544209972110290192136486991790019 |
| 177 | 1.2710274138899515214246518766722806879544209972110290192136486991790019 |
| 178 | 1.2710274138899515214246518766722806879544209972110290192136486991790019 |
| 179 | 1.2710274138899515214246518766722806879544209972110290192136486991790019 |
| 180 | 1.2710274138899515214246518766722806879544209972110290192136486991790019 |