Analytically Interpolating Addition, Multiplication, and Exponentiation

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Abstract

Using the theory of iterated exponentials we interpolate addition, multiplication and exponentiation analytically. We do so in such a manner to respect Goodstein's equation. This solution is found using the modified Bennet operators.

Chapter 1

The modified Bennet operators as objects on the real line.

1.1 Introduction

It's a question as old as analysis. Though, it's one that's never been thought to be solvable. It's more of an absurdist's dream, and something the average mathematician has probably asked, not expecting an answer for. We shall not answer this question, but we'll certainly make headway. What's *in between* addition and multiplication, and then, what's *in between* multiplication and exponentiation?

This is a very difficult question, that tends to leave the average mathematician wondering where you would even start? Where do you look? I've asked this question to pretty much every mathematician I've ever met, in some odd hope that they'd give me some form of insight. Alas, to not much luck. I don't think any mathematician has ever really helped, when it came to this question. So for a very long time I kept it on the back burner.

For a long time I sat on one solution, which was a first order approximation to the correct solution, upon which, I never really knew what to do with the first order solution. I frustrated myself into a cycle, continuously trying to make sense of how to solve for the actual solution from a first order solution. I feel now, after 10 long years of trouble and toil, I can present a discussion, and a solution in limited scenarios. This may not be *the* solution, but it is certainly a solution.

As to set the scene, we have to describe what a solution even is. It is always perfectly possible to interpolate the values $x+y, x\cdot y, x^y$; but that's just cheating. Sure, you can interpolate polynomially, or using something more advanced, but this isn't the answer. What's always to be remembered when talking about addition, multiplication, and exponentiation, is their functional relationship.

$$x + x \cdot y = x \cdot (y+1)$$
$$x \cdot x^y = x^{y+1}$$

So when we go *inbetween* addition and multiplication, we'd expect it to satisfy the same functional relationship *inbetween* multiplication and exponentiation. Simple, no?

We can start by choosing our notation. As much of this is highly non-standardized, we get to choose our notation. To be simple and effective, we choose a slanted *bra-ket* notation. Which then gives us:

$$x \langle s \rangle y$$

Where:

$$x \langle 0 \rangle y = x + y$$
$$x \langle 1 \rangle y = x \cdot y$$
$$x \langle 2 \rangle y = x^y$$

Where we will assume that $0 \leq s \leq 2$ and that $x, y > 1$–at least for now. We will have to be more restrictive in x, y as we progress, but for the moment this is fine. The functional equation we are then asking to solve is,

$$x \langle s \rangle \left(x \langle s+1 \rangle y \right) = x \langle s+1 \rangle (y+1)$$

The *in between* addition and multiplication, satisfies a functional equation with the *in between* multiplication and exponentiation. This is precisely what this functional equation defines. And when I say it's a very difficult equation, I am only beginning to describe the complexity. This is what the author will refer to as Goodstein's equation, in honor of Goodstein who developed a long theory on iterated functions, primitive recursion and what would be known as a hyper-operator [2]. Goodstein's work, and Ackermann's seminal paper [1], on what is known as the Ackermann function, were largely responsible for the mere idea of a hyper-operator–that is, if we disregard Euler's study of tetration.

This enters into our discussion seamlessly. How do we provide an analytic solution? A continuous solution can be found by just pasting together arbitrary solutions. A differentiable solution can be found with a bit more work, same with second differentiable. But really, this becomes a tiresome task. How do you find an analytic solution?

Surprisingly, a solution appears by looking at tetration, or the theory of fractionally iterated exponentials. To do fractionally iterated exponentials though, we have to do a bit of a deep dive into complex dynamics. It seems a tad odd, that to get to $x \langle s \rangle y$, we have to look exclusively at $x \uparrow\uparrow y$ (Knuth's up-arrow notation for tetration), but it is the manner to do.

We begin this paper by constructing a whole swathe of iterated exponentials. We will not have to pull on every thread in the theory, but we'll have have to brush paths with the beginning of the theory.

1.2 The Iterated Exponentials we will need

The bad news first. We will need iterated exponentials, and they are rather difficult to wrap your head around. The good news second. We will need the easiest parts of the theory of iterated exponentials. So, although the theory is very complex, for the limited solutions we will construct, we will use the easiest portions of the theory.

As it's helpful to be expository, we will describe the easiest area in all of iterated exponentials. Let's let $b \in (1, e^{1/e})$. Now, the constant $e^{1/e}$ will appear frequently throughout this study, so it is helpful to give it a name. It's common practise amongst the tetration community to give this the symbol η. It is also typically named for Henryk Trappman. Trappman did much work on the iterated exponential to this base, and for that, we will call this constant Trappman's constant. Where the first case of this nomer, to the author's memory, is Dmitrii Kouznetsov [3].

As the intention of this paper is to discuss a single solution, we won't do a deep dive into the varying types of iterations of the exponential. And as to that, we will give a small recap of the main result from a paper by Trappman and Kouznetsov [4]. To elucidate the affair, we will begin by discussing the case $b = \sqrt{2}$.

The function $f(z) = b^z$ has two fixed points on the real line. It has a geometrically attracting fixed point at 2, and it has a repelling fixed point at 4. Such that:

$$\begin{aligned} f(2) &= \sqrt{2}^2 = 2 \\ f(4) &= \sqrt{2}^4 = 4 \end{aligned}$$

Where:

$$\begin{aligned} 0 < f'(2) &= \log(2) < 1 \\ 1 < f'(4) &= 2\log(2) \end{aligned}$$

To keep things simple, and to not go too far off the deepend, we are going to take what we need from iteration theory without hitting the nitty gritty. And for that, we only care about the fixed point 4. We only care about the repelling fixed point. As described in their joint paper, there exists a function:

$$\exp_{\sqrt{2}}^{\circ s}(z) : \mathbb{C} \times \mathbb{C}/\{z \in \mathbb{R},\, z \leq 4\} \to \mathbb{C}$$

This means, the iteration is holomorphic for all $s \in \mathbb{C}$, and holomorphic in z, so long as $z \notin (-\infty, 4]$. This iteration satisfies the group laws:

$$\exp_{\sqrt{2}}^{\circ s_1}\left(\exp_{\sqrt{2}}^{\circ s_2}(z)\right) = \exp_{\sqrt{2}}^{\circ s_1 + s_2}(z)$$

While simultaneously satisfying:

$$\begin{aligned}
\exp_{\sqrt{2}}^{\circ 1}(z) &= f(z)\\
\exp_{\sqrt{2}}^{\circ 2}(z) &= f(f(z))\\
\exp_{\sqrt{2}}^{\circ 3}(z) &= f(f(f(z)))\\
&\vdots\\
\exp_{\sqrt{2}}^{\circ n}(z) &= f(f(...(n\,\text{times})...f(z)))
\end{aligned}$$

Now, there's nothing too special about the value $b = \sqrt{2}$. For every value $b \in (1, \eta)$, there exists an attracting fixed point on the real line, and a repelling fixed point on the real line. We are again, going to only care about the repelling iteration. So let us write $b^y = y$, while additionally requiring that $\log(y) > 1$, so that the fixed point is repelling.

Then, we can, with no trouble, describe the iteration:

$$\exp_{y^{1/y}}^{\circ s}(z) : \mathbb{C} \times \mathbb{C}/(-\infty, y) \to \mathbb{C}$$

This function will be analytic in y, and holomorphic in z and s. We need a bit more from iteration theory at this point, because from here we have assumed that $x > y > e$, we would like to reduce this to $x, y > e$. This requires us to look at a different domain in s. It is not very difficult.

The function $\log_{y^{1/y}}^{\circ s}(x)$ is analytic for $s \geq 0$ and $y \leq x > e$. And additionally $\log^{\circ s}(x) \geq x$. To be a thorough mathematician, it is helpful to not just blackbox these results. Despite the intensive case study that Kouznetsov and Trappman performed in their paper; it isn't very hard to get this result. That is, if you admit results from complex dynamics. For this, we will cite Milnor [6], and we will present the above result as a theorem.

The first step in getting this result, is performing what is known as a Schröder iteration. This can be done using Koenig's linearization theorem:

Theorem 1.2.1 (Koenig's Linearization Theorem)**.** *If f is holomorphic in the neighborhood of y, such that $f(y) = y$ and $|f'(y)| \neq 0, 1$, then there exists a unique function Ψ, such that $\Psi(y) = 0$ and $\Psi'(y) = 1$–setting $f'(y) = \lambda$:*

$$\Psi(f(z)) = \lambda\Psi(z)$$

From here, we have an iteration, which looks like:

$$f^{\circ s}(z) = \Psi^{-1}(\lambda^s\Psi(z))$$

For repelling fixed points, when $|f'(y)| > 1$, the function $\Psi^{-1}(z)$ is an entire function [6]. And for the cases above, one can show that $\Psi(z)$ is holomorphic for $z \in \mathbb{C}/(-\infty, \omega]$, where ω is the attracting fixed point (for $\sqrt{2}$ this is the value 2). We leave this to the reader, or we ask that you read Kouznetsov and Trappman's paper, which goes into much greater detail than we ever could.

The trouble is on the interval $x \in (\omega, y)$, the function $\exp^{\circ s}(x)$ is holomorphic in a left half plane, for some $\Re(s) < A$. This means that the iterated logarithm is holomorphic for $\Re(s) > -A$, and luckily, $A > 0$.

The next question is of the holomorphy in $b = y^{1/y}$. If the reader were to read Koenig's Linearization Theorem as it's presented in Milnor [6]; then one would notice the solution is holomorphic in the multiplier if we vary it. But not only that, it is holomorphic in the fixed point as we vary it. Holomorphy in y then works for all $y > e$. Where at $y = e$, we are given $\eta = b$, which is a neutral fixed point ($f'(e) = 1$), and Koenig's theorem no longer applies. This then gives us the beginning of our insight into restricting $y > e$.

We summarize all of this in the first real theorems of our paper.

Theorem 1.2.2. *Let $y > e$, let $x > y$, and let $s \in \mathbb{R}$; there exists an analytic function in all variables:*

$$\exp^{\circ s}_{y^{1/y}}(x)$$

Such that $\exp^{\circ s}_{y^{1/y}}(x) > y$.

Proof. Per Trapmann & Kouznetsov [4]. □

Theorem 1.2.3. *Let $x, y > e$, and let $s > 0$; there exists an analytic function in all variables:*

$$\log^{\circ s}_{y^{1/y}}(x)$$

Such that $\log^{\circ s}_{y^{1/y}}(x) > x$ for $x < y$, and $\log^{\circ s}_{y^{1/y}}(x) < x$ for $x > y$.

Proof. This extends a bit further from Trapmann and Kouznetsov [4]. □

These two theorems epitomize all we need to know about iterated exponentials. It leads us perfectly to the next section.

1.3 The first order interpolation

It takes a lot of explanation to talk about the next result. This result dates back at least 10 years within my own work. But as I never felt a use for it, it only exists as an investigation on The Tetration Forum [7]. This is a clever manner of interpolating addition, multiplication, and exponentiation using iterated exponentials. It was discovered by myself and Sheldon Levenstein, back when I was 19 years old and knew next to nothing about mathematics. Sheldon was instantly shut down when he realized it was only a first order interpolation. Though, truly, this expansion was birthed from a nubile brain playing with his program fatou.gp [5].

Let us define a preliminary solution to $x \langle s \rangle y$, and call it $x\,[s]\,y$. Our solution will be a small error term which corrects the first order to our solution. For that, we remind the reader we assume that $x, y > e$.

$$x\,[s]\,y = \exp_{y^{1/y}}^{\circ s}\left(\log_{y^{1/y}}^{\circ s}(x) + y\right)$$

To ensure this is a legal expression, we check that $\log_{y^{1/y}}^{\circ s}(x) > x > e$ when $y > x > e$; and $e < y < \log_{y^{1/y}}^{\circ s}(x) < x$. So that the interior of the iterated exponential $\log_{y^{1/y}}^{\circ s}(x) + y > y + e$, which means the iterated exponential is fine.

We will perform our first check, that it interpolates the correct values. But to do that, we remind the reader that:

$$\log^{\circ s} = \exp^{\circ -s}$$

Again, by Theorem 1.2.2 and Theorem 1.2.3:

$$x\,[s]\,y$$

is analytic in all variables involved. It does the job of interpolation just as well. But this may be a bit trickier to visualize.

The function:

$$\begin{aligned}
x\,[0]\,y &= \exp^{\circ 0}\left(\log^{\circ 0}(x) + y\right)\\
&\log^{\circ 0}(x) = x\\
&\exp^{\circ 0}(x+y) = x + y\\
&\text{Therefore}\\
x\,[0]\,y &= x + y
\end{aligned}$$

Now to the second interpolation point:

$$
\begin{aligned}
x\,[1]\,y &= \exp^{\circ 1}_{y^{1/y}}\left(\log^{\circ 1}_{y^{1/y}}(x) + y\right)\\
&\log^{\circ 1}_{y^{1/y}}(x) = y\log(x)/\log(y)\\
&\exp^{\circ 1}_{y^{1/y}}(w) = e^{\log(y)w/y}\\
&\text{Therefore}\\
x\,[1]\,y &= e^{\log(y)(y\log(x)/\log(y)+y)/y}\\
&= e^{\log(x)+\log(y)}\\
&= xy
\end{aligned}
$$

As for the third interpolation point, we have to pay strict attention to y. There's a reason we are choosing the iterated exponentials we are choosing. For this, we introduce the formula for exponentiation:

$$
\begin{aligned}
x^y = x\,[2]\,y &= \exp^{\circ 2}_{y^{1/y}}\left(\log^{\circ 2}_{y^{1/y}}(x) + y\right)\\
&\exp^{\circ 2}_{y^{1/y}}\left(\log^{\circ 2}_{y^{1/y}}(x) + y\right) = \exp_{y^{1/y}}\left(\log_{y^{1/y}}(x)\cdot y\right)\\
&\text{Therefore}\\
x\,[2]\,y &= e^{\log(y)\left(y^2\log(x)/\log(y)\right)/y}\\
&= e^{\log(x)y}\\
&= x^y
\end{aligned}
$$

You can see from these equations alone, that we're already kind of going *in between* addition, multiplication and exponentiation. And we're using an interpolation method that depends on the exponential/logarithm's functional equation. But we are additionally paying extra care, such that this is an analytic interpolation about a fixed point. And not just analytic, it is conducive to a Goodstein functional equation. It's only off by a couple of digits.

This is what Vittorio and I have dubbed the modified Bennet operators. They take their name from Bennet's commutative operators. These operators are home to group theory typically, and look like:

$$x \oplus_n y = \exp^{\circ n}\left(\log^{\circ n}(x) + \log^{\circ n}(y)\right)$$

They are a commutative infinite chain of operators that have the distributive property for $\oplus_{n+1}$ over $\oplus_n$, while equalling $+$ and $\cdot$ for $n = 0$ and $n = 1$ respectively.

By this, we choose $y = 3 > e$ and $x = 4 > y$. Such we can draw the first order approximation $4\,[s]\,3$ of the actual solution $4\,\langle s\rangle\,3$. The first order approximation is seen in Figure 1.1. A comparison between $4\,[s]\,(4\,[s+1]\,3) = 4\,[s+1]\,4$ is displayed in Figure 1.2.

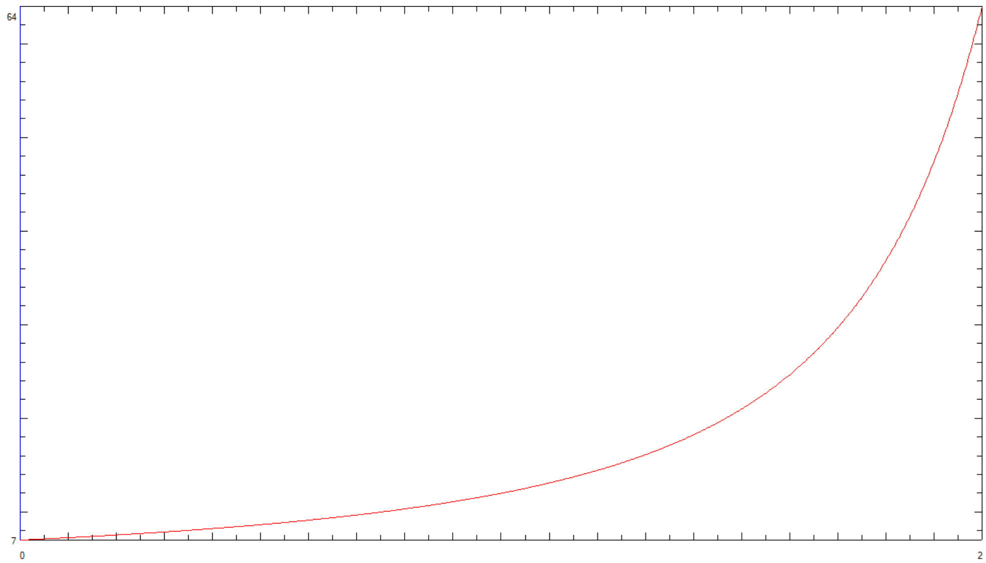

Figure 1.1: The function $4\,[s]\,3$ over the domain $0 \leq s \leq 2$.

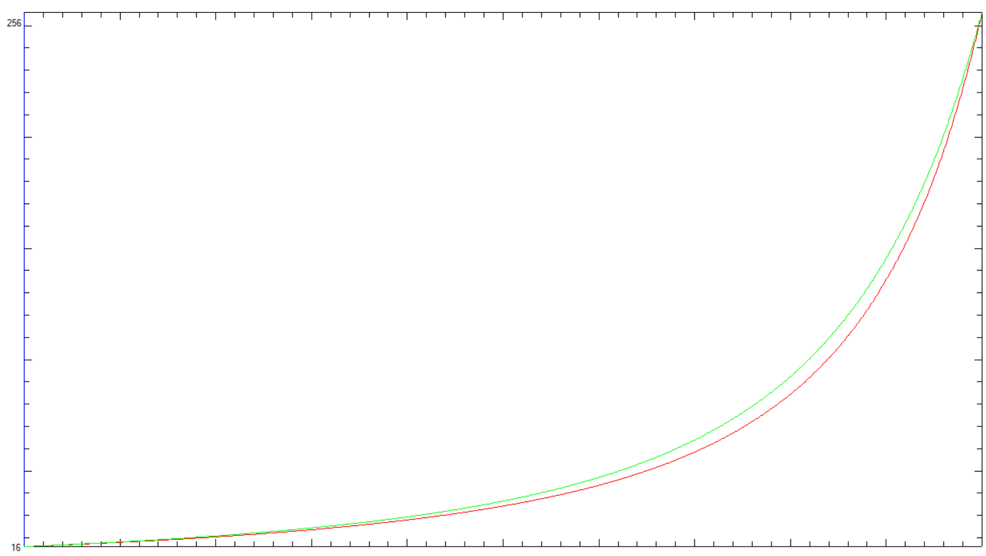

Figure 1.2: The function $4\,[s]\,(4\,[s+1]\,3)$ plotted next to $4\,[s+1]\,4$, over the interval $s \in [0, 1]$.

We can solve for $\gamma_0, \gamma_1, \gamma_2$, as a function function in x, y, such that:

$$x\,\langle s\rangle\,y = (x\,[s]\,y) + \gamma_0 \cdot s + \mathcal{O}(s^2)$$
$$x\,\langle s\rangle\,y = (x\,[s]\,y) + \gamma_1 \cdot (s-1) + \mathcal{O}((s-1)^2)$$
$$x\,\langle s\rangle\,y = (x\,[s]\,y) + \gamma_2 \cdot (s-2) + \mathcal{O}((s-2)^2)$$

This gives us a good strong look at the initial first order approximation. From here, we have to dive much much deeper.

1.4 The φ argument

The remainder of this paper will focus on a much deeper type of analysis. We will brush paths with Riemann surfaces, and with evolving surfaces. To begin our study, we will have to introduce a third notation, and describe a Riemann surface that evolves with the variables x, y and s. As the author is avoiding deep problems within complex analysis for this paper, it is perfectly safe to consider this as a surface, and drop the Riemann part. But it is much more enlightening if you remember that it is a Riemann surface at its core.

Denote the following:

$$x\,\langle s\rangle_\varphi\,y = \exp_{y^{1/y}}^{\circ s}\left(\log_{y^{1/y}}^{\circ s}(x) + y + \varphi\right)$$

Where we're given the equivalence:

$$x\,\langle s\rangle_\varphi\,y = (x\,[s]\,y) + \mu \cdot \varphi + \mathcal{O}(\varphi^2)$$

We can always be sure that $\mu \neq 0$, and we can be certain that $\mu > 0$. We will provide a proof for this later in the paper. Momentarily, we would like for the reader to take it as gospel.

To commence a difficult journey, we construct a surface in $\mathbb{R}^3$, which is of dimension $\mathbb{R}^2$. To do so, we begin by fixing $0 \leq s \leq 1$ and $x, y > e$. Then we describe a surface:

$$F(\varphi_1, \varphi_2, \varphi_3) = x\,\langle s\rangle_{\varphi_1}\,(x\,\langle s\rangle_{\varphi_2}\,y) - x\,\langle s\rangle_{\varphi_3}\,(y+1) = 0$$

The partial derivatives in all variables are non-zero, which again, we will prove later in this paper–take it as gospel momentarily. This means that:

$$\frac{\partial F}{\partial \varphi_i} \neq 0 \text{ for } i = 1, 2, 3$$

If we choose a single point $\boldsymbol{\varphi}^0 = (\varphi_1^0, \varphi_2^0, \varphi_3^0)$ which is a solution to the above equation; then there exists points within in a neighborhood $||\boldsymbol{\varphi} - \boldsymbol{\varphi}^0||_2 < \delta$, which continue to satisfy this equation. Here we use the standard Euclidean norm on $\mathbb{R}^3$. This is enough for us to confirm that F describes a surface in $\mathbb{R}^3$.

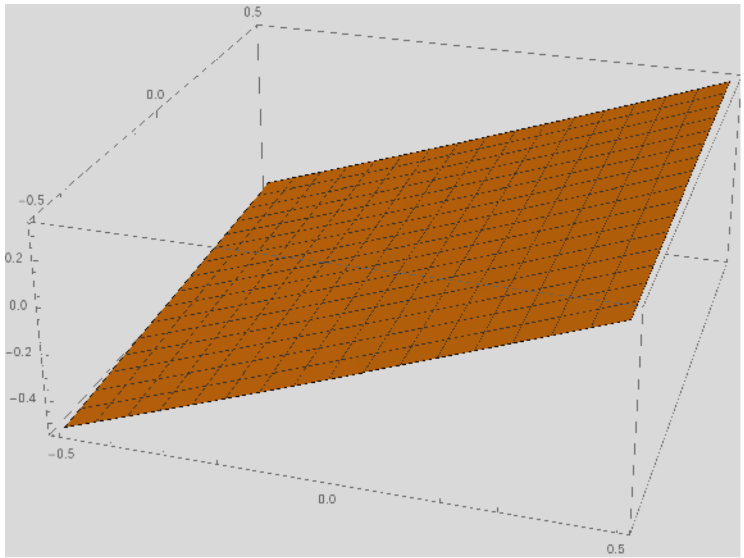

Figure 1.3: The surface F for $s = 0.5$ and $x, y = 3$. It is graphed over the domain $-0.5 \leq \varphi_1, \varphi_2 \leq 0.5$. Notice it is very planar.

Now, as an important reminder, this equation will begin to break down once:

$$\log_{y^{1/y}}^{\circ s}(x) + y + \varphi \leq y$$

We are lucky though, and φ remains relatively small throughout our calculations. And in order for this to happen we would have to have $\varphi \leq -e$. So for our purposes we can add a limiter at these values, if either of the φ_i, exceed this limit. It's serendipitous for us that this won't happen for the values we are concerning ourselves.

Another way to view this as a surface, is to create two free variables φ_1 and φ_2, such that:

$$\varphi_3 = \log_{(y+1)^{1/(y+1)}}^{\circ s+1}(x \langle s \rangle_{\varphi_1} (x \langle s+1 \rangle_{\varphi_2} y)) - y - 1 - \log_{(y+1)^{1/(y+1)}}^{\circ s+1}(x)$$

And now you simply graph the surface across $\varphi_1, \varphi_2 > -e$, and let φ_3 be the function value above. We've included a graph of one of these surfaces in Figure 1.3. The result is very planar, we can see that the relation between these variables is fairly linear. This is very helpful for our study.

We are searching for a single value on this surface, which provides the correct answer. This value, we will refer to as $\boldsymbol{\varphi}$. Then, we can watch the evolution of this value by varying it in (x, y, s). This enters into the very difficult problem, of what function $\boldsymbol{\varphi}$ is the correct function. We can make arbitrary functions which satisfy the equation.

For example, set $\varphi_1, \varphi_2 = 0.1 \cdot s(1-s)$, then φ_3 is given as above. And we'll have a working solution $\boldsymbol{\varphi}$, which satisfies Goodstein's equation for local values in x, y. The trouble is, this will not be an operator on x, y, s. For this we move into murkier waters.

1.5 The Restrictions to our equation

We have a plethora of candidate functions $\boldsymbol{\varphi}$, but there's only one that will work. And to find this one, we need to understand the exact restrictions we need on the surface F to isolate a single point. In order to do this, we have to talk about how the surface evolves. For this, we require talking about $\boldsymbol{\varphi}$ in y and in s. The variable x, does not play much of a role for the remainder of this paper as it is fixed in Goodstein's equation.

If we describe a path:

$$\boldsymbol{\varphi}(y,s) = (\varphi_1(y,s), \varphi_2(y,s), \varphi_3(y,s))$$

On the evolving surface:

$$F(y, s, \varphi_1, \varphi_2, \varphi_3) = 0$$

Then the restrictions we require are specifically tailored to make $x \langle s \rangle_{\boldsymbol{\varphi}} y$ into an operator. The first restriction is pretty simple to understand. We will write it, and then we'll give an explanation.

$$\varphi_2(y+1, s) = \varphi_3(y, s)$$

If we were to write:

$$x \langle s \rangle_{\boldsymbol{\varphi}(y,s)} y$$

And assign this as an operator depending only on x, y, s and no additional inputs, then we would like for:

$$x \langle s+1 \rangle_{\varphi_2} y$$

And

$$x \langle s+1 \rangle_{\varphi_3} y + 1$$

To be the same function, where we've only shifted y forward by 1 in the second equation. So we can assume that they don't vary dependence on s, but when you move φ_2 forward in y by 1, we get the value φ_3. This reduces the set of possible values for $(\varphi_1, \varphi_2, \varphi_3)$ into a one dimensional curve in $\mathbb{R}^3$.

The second restriction is more difficult to explain. We only have the surface as a defined thing for $0 \leq s \leq 1$, and the second restriction is localized to $s = 1$. We write it out, and we qualify what we mean by the restriction:

$$\varphi_1(y,1) = \varphi_2(x\,\langle 2\rangle_{\varphi_2(y,1)}\,y, 0)$$

To make sense of this, we have to clarify, that where you see constants here, we are asking that the Taylor expansions also agree. This becomes a very difficult problem to ensure equality. A pointwise agreement would ensure continuity, but since we are asking for analycity, this becomes more difficult. But we can write it out plainly as:

$$\frac{d^k}{ds^k}\Big|_{s=0}\varphi_1(y,1+s) = \frac{d^k}{ds^k}\Big|_{s=0}\varphi_2(x\,\langle 2+s\rangle_{\varphi_2(y,1+s)}\,y, s) \text{ for } k \geq 0$$

For the moment, if we assume this system of equations is solvable, then we have our solution for inbetween multiplication and exponentiation. It is given as:

$$x\,\langle 1+s\rangle\,y = x\,\langle 1+s\rangle_{\varphi_2(y,s)}\,y$$

From here, to construct in between addition and multiplication we need only do the following. Let $f(y,s) = x\,\langle 1+s\rangle\,y$ and let $f^{-1}(y,s)$ be the functional inverse in y. We get the lower order operators:

$$x\,\langle s\rangle\,y = f(1 + f^{-1}(y,s), s)$$

Where analycity is guaranteed at $s = 1$ because of the above equality between Taylor coefficients.

1.6 Vittorio's Family

To begin this section, we're going to reapproach the question at hand. The last few sections have helped us feel out a solution, but there's nothing concrete and in our hands. We are going to continue in this spirit, and leave the heavy lifting for later in this paper. But this represents the second solution criterion, which will help us find a unique solution.

This criterion is not Vittorio's, but a limit which will be very valuables hinges on this criterion. The limit is Vittorio's, but the criterion for it, is something subtle–but little different than Goodstein's operations. We can start flatly by describing a notational result:

$$x\,\langle s\rangle\,y = x\,\langle s+1\rangle\,\big((x\,\langle s+1\rangle^{-1}\,y) + 1\big)$$

This means, if $f(s,y) = x\,\langle s\rangle\,y$, and f^{-1} is the inverse in y, then any solution to Goodstein satisfies:

$$f(s,y) = f(s+1, f^{-1}(s+1,y) + 1)$$

Vittorio's limit, is essentially, a Bolzano-Weierstrass limit about the family of functions in the area of each solution. Which we can say $f(s,y) \in \mathcal{F}$, and a sequence of said f, which satisfy:

$$f_n(s,y) = f_{n-1}(s+1, f_{n-1}^{-1}(s+1,y)+1)$$

We do not want this exact form of the equation. Instead we what the equation in this form:

$$f^{-1}(s+1, f(s,y)) = f^{-1}(s+1,y)+1$$

Which is to mean that we want this equation in Abel's form. This helps us, as what we are really looking for is solutions to Abel's equation, but additionally, in Abel's form the error terms are rather small. This is very helpful in describing what a net of solutions looks like.

We can now introduce the family of functions we want to describe and find a limit of. It's a bit wordy, but it describes the basis for what will be our heavy lifting portion of this paper.

Definition 1.6.1 (Vittorio's Family)**.** Let $\mathcal{F}$ be the family of functions $f : [0,2] \times (e,\infty) \to (e,\infty)$ such that:

- f is analytic.
- $f(0,y) = x+y$ for $x > e$
- $f(1,y) = x \cdot y$
- $f(2,y) = x^y$
- $f(s,y) = O(y)$ for all $0 \leq s \leq 1$
- $|f^{-1}(s+1, f(s,y)) - f^{-1}(s+1,y) - 1| = o(y^\epsilon)$ for all $0 \leq s \leq 1,$ all $\epsilon > 0$
- $|f(s,y) - x\,[s]\,y| = o(y^\epsilon)$ for all $0 \leq s \leq 1,$ all $\epsilon > 0$

This family serves as a net we are casting about the potential solutions, and the actual solution we want. This family is rather broad in its scope. If I take $f, g \in \mathcal{F}_\epsilon$, then:

$$\frac{f(s,y) - g(s,y)}{y^\epsilon} \to 0 \text{ as } y \to \infty$$

We can define a distance function on this family given as:

$$||f-g||_{\epsilon,Y} = \sup_{s\in[0,1]} \sup_{y>Y} |f(s,y) - g(s,y)|/|y|^\epsilon$$

Under this distance function, we aim to show that $\mathcal{F}$ is a normal family. This isn't too intricate, and will be handled in the next section more completely, but it appears rather clearly, as these solutions will be locally bounded in this norm.

We would like to speak briefly about the last two conditions in this family. And, a result which is needed to even verify the existence of this family. Technically, we have made a *faux pas* in this definition. We have partially assumed that $x\,[s]\,y$ is in this family. We can remedy this with a couple observations; which we'll put in the next section.

More importantly, we'd like to qualify the penultimate and last condition. If we momentarily ask that the reader suspend belief, and assume $x\,[s]\,y$ satisfies the penultimate condition (it does), then we can note that $x\,[s]\,y$ is in Vittorio's family. We ask that every other solution satisfies the penultimate condition, while additionally being relatively close to $x\,[s]\,y$.

As we have not strayed too far from the goal laid out in earlier sections, we are trying to 'correct' $x\,[s]\,y$ such that it satisfies Goodstein's equation. We are assuming, and in many ways, wildly guessing that the solution to Goodstein's equation exists within this net. Somewhere around $x\,[s]\,y$, we have $x\,\langle s\rangle\,y$. This family describes the bare-minimum this solution must satisfy.

This will be handled later, and is the centerpiece of the next section. We are trying to set up a limit process, which lends itself to this notation. We want this construction to be valid, while the previous first order equation from the last section is satisfied. We're juggling many ideas, and many possible solutions at the moment–allez-y.

1.7 Vittorio's Family is Normal

The goal of this section is to show that the family $\mathcal{F}$ from Definition 1.6.1 is normal when using the norm $||f||_{\epsilon,Y}$ for some value Y. To remind the reader of normality. We can explain the goal.

Does any sequence $f_n \in \mathcal{F}$ have a converging subsequence? By which–for all $\rho > 0$ there exists K, such for $k, j > K$:

$$||f_{n_k} - f_{n_j}||_{\epsilon,Y} < \rho$$

To show this is the goal of this section. By, showing that Vittorio's Family 1.6.1 is Normal, we open the gates to advanced calculus.

On commence par, the inclusion of $x\,[s]\,y \in \mathcal{F}$. From here, normality becomes a story of moral conclusion. Vittorio's family is normal by the simple inclusion of $x\,[s]\,y$. To get there though, it's important to explain how to evaluate the inverse of $x\,[s+1]\,y$, the function $x\,[s+1]^{-1}\,y$.

In order to do this, we solve an iterative formula. If I write:

$$y = x\,[s+1]\,\omega = \exp^{\circ s}_{\omega^{1/\omega}}\left(\log^{\circ s}_{\omega^{1/\omega}}(x)\cdot\omega\right)$$

And we define the function:

$$f(w) = \frac{\log^{\circ s}_{w^{1/w}}(y)}{\log^{\circ s}_{w^{1/w}}(x)}$$

Then the value ω is a fixed point, $f(\omega) = \omega$. Furthermore, it is an attracting fixed point. And additionally, one can put y within the basin of this fixed point. Such that:

$$\lim_{n\to\infty} f^{\circ n}(y) = \omega$$

We do not need to elaborate much on this construction, but it explains the quickest manner the author could find at solving for the function $\omega = x\,[s+1]^{-1}\,y$. This is solely helpful for visualization purposes, as the author sees it. In Figure 1.5 we've included an example of one of these functions.

But nonetheless, it is important to show, that for $w > \omega$, this process converges geometrically. This lets us control the inverse in a native manner. It shows a separate limit structure intrinsic to the modified Bennet operators.

Lemma 1.7.1. *Let* $f(w) = \frac{\log^{\circ s}_{w^{1/w}}(y)}{\log^{\circ s}_{w^{1/w}}(x)}$*, then there exists a fixed point* ω*, such that for all* $w > \omega$ *and sufficiently large* $y > Y$*:*

$$\lim_{n\to\infty} f^{\circ n}(w) = \omega$$

Additionally, $f'(\omega) = \lambda < 1$.

Proof. To begin, we know the fixed point ω exists for $y > Y$ for large enough Y because this fixed point is the value such that:

$$x\,[s+1]\,\omega = y$$

And the function $x\,[s+1]\,y$ is invertible for large enough $y > Y$. Therefore, the only question, is if the iterates converge to the solution. To explain this, all we need to do is look at:

$$f'(\omega) = \lambda$$

Which determines if the fixed point is geometrically attracting or not. But, to make a quicker guess, we can identify that:

$$f(w) < w$$

Therefore $f^{\circ n}(w)$ is a decreasing sequence which must converge to ω. The function $f(w)$ is increasing in w, and tends to infinity. Its growth is $\mathcal{O}(\log(w))$, but sussing this out is a tad difficult. We've included a picture of one such f in Figure 1.4

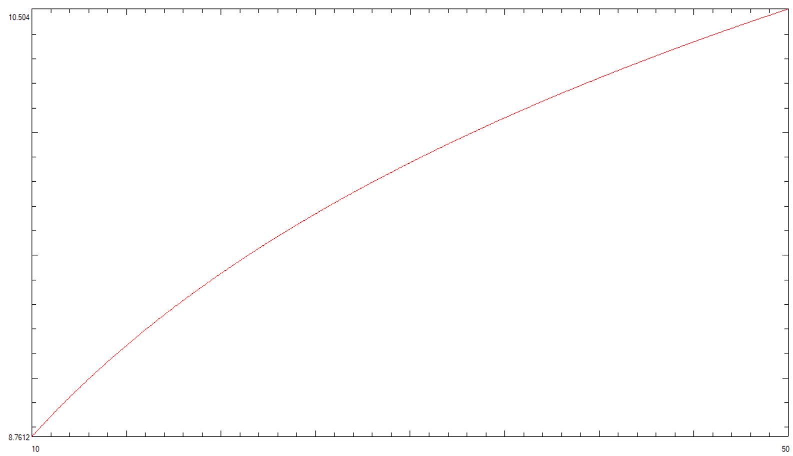

Figure 1.4: This is the function $f(w) = \frac{\log^{\circ 0.5}_{w^{1/w}}(100)}{\log^{\circ 0.5}_{w^{1/w}}(3)}$ graphed over the domain $w \in [10, 50]$.

Assume that $w > \omega$, then $y > \omega$ (necessary as it's the inverse value), so that $\log^{\circ s}_{\omega^{1/\omega}}(y) < y$, and $\log^{\circ s}_{\omega^{1/\omega}}(x) \geq \min(x, \omega)$ (depending on whether $x \leq \omega$ or not). Therefore:

$$\omega \leq \frac{y}{\min(x, \omega)}$$

And for the general case, assume that $w > y$, then:

$$f(w) < \frac{\log(y)w}{\log(w)\min(x, w)}$$

As we are limiting $w \to \infty$ it eventually surpasses x, and therefore $\log^{\circ s}_{w^{1/w}}(x) > x$. And therefore a more feasible estimate is:

$$f(w) < \frac{\log(y)w}{\log(w)x}$$

So as this function grows it looks like $o(w)$. Now assume that there was another fixed point ω' such that:

$$f(\omega') = \omega'$$

Well this would imply that there is a second inverse to $x\,[s+1]\,y$, which is preposterous because it is a monotone increasing function that grows faster than $x \cdot y$. Therefore we must have $f(w) < w$. This ensures that $f^{\circ n}(w) \to \omega$ for all $w > \omega$.

Since the fixedpoint is monotone increasing as well, we must have that $f'(\omega) \to 0$, and so therefore, for large enough $y > Y$ we must have $f'(\omega) = \lambda < 1$, and the fixedpoint is geometrically attracting. $\square$

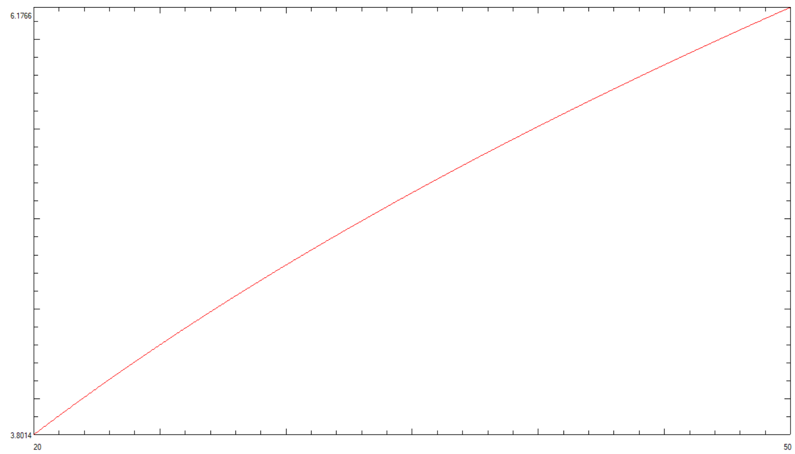

Figure 1.5: This is a graph over $20 \leq y \leq 50$ of the function $3\,[1.5]^{-1}\,y$. The algorithm is done through the iterative procedure of the fixed point $f(\omega) = \omega$.

We are approaching two very difficult theorems now. We have to show that $x\,[s]\,y \in \mathcal{F}$, and additionally we must show that $\mathcal{F}$ is normal. The second result becomes a self-fulfilling prophecy, once we show that $x\,[s]\,y \in \mathcal{F}$. Normality becomes a second thought, corollary from this.

Theorem 1.7.2. *The function $f(s, y) = x\,[s]\,y \in \mathcal{F}$ for $x > e$.*

Proof. We need to show that $x\,[s]\,y = O(y)$ for $0 \leq s \leq 1$, and additionally show that $\big(x\,[s+1]^{-1}\,(x\,[s]\,y)\big) - x\,[s+1]^{-1}\,(y) - 1 = o(y^{\epsilon})$. The rest of the criterion for belonging to $\mathcal{F}$ of Definition 1.6.1 are self explanatory. The first claim follows because $x\,[0]\,y = x + y$ and $x\,[1]\,y = x \cdot y$, and $x\,[s]\,y$ is monotonely increasing in s.

The second claim is more difficult to address. If we return to the function:

$$f(s, y, w) = \frac{\log_{w^{1/w}}^{\circ s}(y)}{\log_{w^{1/w}}^{\circ s}(x)}$$

Then iterates of $f(s, y, w)$ converge to $\omega(s, y)$. We can set w to any value greater than ω. The value of $\omega(s, x\,[s]\,y) < y$ so we can start the iteration from y. This means that:

$$f(s, x\,[s]\,y, y) = \frac{\log_{y^{1/y}}^{\circ s}(x\,[s]\,y)}{\log_{y^{1/y}}^{\circ s}(x)}$$

But this just looks like:

$$\begin{aligned} f(s, x\,[s]\,y, y) &= \frac{\log^{\circ s}_{y^{1/y}}\left(\exp^{\circ s}_{y^{1/y}}\left(\log^{\circ s}_{y^{1/y}}(x) + y\right)\right)}{\log^{\circ s}_{y^{1/y}}(x)} \\ &= \frac{\log^{\circ s}_{y^{1/y}}(x) + y}{\log^{\circ s}_{y^{1/y}}(x)} \\ &= 1 + \frac{y}{\log^{\circ s}_{y^{1/y}}(x)} \end{aligned}$$

Which is exactly:

$$f(s, x\,[s]\,y, y) = 1 + f(s, y, y)$$

So, we are asking that the further orbits stay within each other. We can write this more clearly by assigning:

$$f_2(w) = f(s, x\,[s]\,y, w)$$

Then there exists a fixed point:

$$f_2(\omega_2) = \omega_2$$

We wish to show that:

$$\omega_2 \approx \omega + 1$$

What's helpful to remember is that $f^{\circ n}(y) = \omega + \mathcal{O}(q^n)$ and similarly for $f_2^{\circ n}(y) = \omega_2 + \mathcal{O}(q^n)$ for some $0 < q < 1$. So, as we iterate either function they are within difference of each other geometrically. Therefore:

$$\omega_2 + \mathcal{O}(q) = 1 + \omega + \mathcal{O}(q)$$

As $\omega \to \infty$ (which equates to $y \to \infty$) as we detailed in Lemma 1.7.1, we know that $f'(\omega) \to 0$, and thereby we can choose $q \to 0$ as $\omega \to \infty$. So therein, for large enough $y > Y$, we must have the result that:

$$\omega_2 = 1 + \omega + o(y^\epsilon)$$

□

Now that we know $x\,[s]\,y$ is in $\mathcal{F}$, we can write every element of $f \in \mathcal{F}$ as:

$$f(s, y) = x\,[s]\,y + h(s, y) = x\,[s]\,y + o(y^\epsilon)$$

To begin, we're going to note that $f(s, y) \in \mathcal{F}$ is $\mathcal{O}(y)$ for $s \in [0, 1]$. This is by construction. Thereby,

$$|f(s, y)|/y = |f(s+1, f^{-1}(s+1, y) + 1)|/y + o(y^{\epsilon - 1})$$

By which we can say:

$$\left|\left|\frac{f(s,y)-f(s+1,f^{-1}(s+1,y)+1)}{y}\right|\right|_{\epsilon,Y}=O(Y^{-1})$$

We should choose Y appropriately large, such that this value stays relatively small, but this identity tells us that we can shrink the value of this norm, dependent on Y. We will set a value as we progress. But this is enough to produce that Vittorio's Family is normal, for large enough Y under the norm $||...||_{\epsilon,Y}$.

Theorem 1.7.3 (Vittorio's Family is Normal)**.** *Vittorio's family 1.6.1 is a Normal family for large enough $y > Y$, and arbitrary $\epsilon > 0$, under the norm:*

$$||h||_{\epsilon,Y}=\sup_{0\leq s\leq 1}\sup_{y>Y}\left|\frac{h(s,y)}{y^\epsilon}\right|$$

1.8 The φ solution

Before proving the solution, the author would like to pull the solution from a hat. This becomes a very difficult idea to convey. And so it is helpful to entirely explain the solution before we actually prove it's the answer. We begin by declaring the following system of equations. Let $\varphi(s,y)$ be analytic for $0 \leq s \leq 1$. Let's assume that:

$$\begin{aligned}\varphi(0,y)&=0=\varphi(1,y)\\ \varphi(s,y)&=\varphi(s,x\langle s\rangle_{\varphi(s,y)}y)\end{aligned}$$

Now let's define the Abel solution α_φ to the equation:

$$\alpha_\varphi(x\langle s\rangle_\varphi y)=\alpha_\varphi(y)+1$$

Then observe a very complicated self referential relationship.

$$\begin{aligned}g(s,y)&=x\langle s+1\rangle_{\varphi(s,y)}\alpha_{\varphi(s,y)}(s,y)\\ g(s,x\langle s\rangle_{\varphi(s,y)}y)&=x\langle s+1\rangle_{\varphi(s,y)}\left(\alpha_{\varphi(s,y)}(s,y)+1\right)\\ g(s,x\langle s\rangle_{\varphi(s,y)}x\langle s\rangle_{\varphi(s,y)}y)&=x\langle s+1\rangle_{\varphi(s,y)}\left(\alpha_{\varphi(s,y)}(s,y)+2\right)\\ &\vdots\\ g(s,x\langle s\rangle^n_{\varphi(s,y)}y)&=x\langle s+1\rangle_{\varphi(s,y)}\left(\alpha_{\varphi(s,y)}(s,y)+n\right)\end{aligned}$$

Well as it turns out, α is the inverse to $x\,\langle s+1\rangle_{\varphi(s,y)}\,y$, so $g(s,y)=y$. This then shows that:

$$x\,\langle s\rangle_{\varphi(s,y)}\,y = x\,\langle s+1\rangle_{\varphi(s,y)}\,\left(\alpha_{\varphi(s,y)}(s,y)+1\right)$$

And this is our solution. Which translates into, if $f(s,y) = x\,\langle s\rangle_{\varphi(s,y)}\,y$ as the condition:

$$f(s,y) = f(s+1, f^{-1}(s+1,y)+1)$$

To begin this study, we have to recall that everything is a free variable. By which, φ especially is a free variable. If we denote:

$$x\,\langle s+1\rangle_{\varphi}\,\alpha_{\varphi}(y) = y$$

Every variable here is free, so long as $x, y > e$ and $\varphi > -e$. Then, the center point of the construction is looking into the function $H(\varphi)$ such that:

$$H(s,y,\varphi) = \alpha_{\varphi}\left(x\,\langle s\rangle_{\varphi}\,y\right) - \alpha_{\varphi}(y) - 1$$

The function $\varphi(s,y)$ is precisely the value:

$$H(s,y,\varphi(s,y)) = 0$$

Which boils into finding a root of the above equation. The root always exists... though to show this is rather complicated. The theory we have presented so far, is to argue that there is a net of solutions in which the solution should exist. Which is to mean, there is some φ where this solution exists. By which, the solution to this problem is solved by finding a zero. Full. Stop.

We are going to spend the remainder of this paper proving this, and justifying this. We are going to prove as much as we can of an interpolation. To begin, we argue that, the solution always exists for $|s| < \delta$ and $|s-1| < \delta$ for some $\delta > 0$.

In order to do this, we need to only show that $\frac{\partial}{\partial\varphi}H \neq 0$ for these two extremes. Then the implicit function theorem takes care of the rest. We will spend the remainder of this section focusing on the implicit solution about $s \approx 0$.

We note that:

$$\begin{aligned}
&x\,\langle 0\rangle\,_{\varphi}y = x + y + \varphi\\
&x\,\langle 1\rangle\,_{\varphi}y = x \cdot y e^{\varphi\log(y)/y} \approx x \cdot y\left(1 + \varphi\log(y)/y + O(\varphi\log(y)/y)^2\right)
\end{aligned}$$

Where:

$$\begin{aligned}
\alpha_{\varphi}(0,y) &= \frac{y}{x\left(1+\varphi\log(y)/y + O(\varphi\log(y)/y)^2\right)}\\
\alpha_{\varphi}(0,x+y+\varphi) &= \alpha_{\varphi}(0,y)+1
\end{aligned}$$

The solution of which is $\varphi = 0$, but in differentiating this solution, we see that $\frac{\partial H}{\partial \varphi}(s, y, \varphi)\Big|_{s=0,\varphi=0} \neq 0$. This suffices to show an implicit solution exists near 0. We write this quickly.

Theorem 1.8.1. *Let $\alpha_\varphi(s, y)$ be the inverse to $x \langle s+1 \rangle_\varphi y$. Let:*

$$H(s, y, \varphi) = \alpha_\varphi(s, x \langle s \rangle_\varphi y) - \alpha_\varphi(s, y) - 1$$

Then:

$$\frac{\partial H}{\partial \varphi}(s, y, \varphi)\Big|_{s=0,\varphi=0} \neq 0$$

For large enough Y, with $y > Y$.

Proof.

$$\begin{aligned}
&\alpha_\varphi(0, x \langle 0 \rangle_\varphi y) - \alpha_\varphi(0, y) - 1 = \\
&= \frac{y + x + \varphi}{x\,(1 + \varphi \log(y)/y + O(\varphi \log(y)/y)^2)} - \frac{y}{x\,(1 + \varphi \log(y)/y + O(\varphi \log(y)/y)^2)} - 1 \\
&= \frac{x + \varphi}{x\,(1 + \varphi \log(y)/y + O(\varphi \log(y)/y)^2)} - 1
\end{aligned}$$

From this equation, we take the derivative in φ and set it to 0 which gives us:

$$\begin{aligned}
\frac{x + \varphi}{x\,(1 + \varphi \log(y)/y + O(\varphi \log(y)/y)^2)} - 1 &= \frac{1}{1 + \varphi \log(y)/y} - 1 + \frac{\varphi}{x} + O(\varphi^2) \\
&= \left(\frac{1}{x} - \log(y)/y\right)\varphi + O(\varphi^2)
\end{aligned}$$

Therefore, these functions have a nonzero derivative at $\varphi = 0$, provided that $y > Y$ is large enough. $\square$

Corollary 1.8.2. There exists an analytic function $\varphi(s, y)$ for $y > Y$ large enough, and $|s| < \delta$, such that:

$$\alpha_{\varphi(s,y)}(s, x \langle s \rangle_{\varphi(s,y)} y) = \alpha_{\varphi(s,y)}(s, y) + 1$$

This does not guarantee a solution yet. For that, we need to show that $\varphi(s, x \langle s \rangle_{\varphi(s,y)} y) = \varphi(s, y)$. This proves to be a fairly difficult task. It is difficult as it relates a multivariable equation. We want to encompass the following theorem to work for the general case $0 \leq s \leq 1$, but in truth, we are only going to show it for $|s| < \delta$, as this is the only place so far we are guaranteed a solution.

Theorem 1.8.3. *Any solution $\varphi(s,y)$ to the equation $H(s,y,\varphi(s,y)) = 0$ satisfies:*

$$\varphi(s, x \langle s \rangle_{\varphi(s,y)} y) = \varphi(s,y)$$

Proof. We are going to fix the value $\varphi_0 = \varphi(s,y)$. Which solves the equation:

$$\alpha_{\varphi_0}(s, x \langle s \rangle_{\varphi_0} y) = \alpha_{\varphi_0}(s,y) + 1$$

We want to consider the orbits of $x \langle s \rangle_{\varphi_0} y$ now:

$$\begin{aligned}
x\langle s\rangle_{\varphi_0}^0 y &= y \\
x\langle s\rangle_{\varphi_0}^1 y &= x \langle s \rangle_{\varphi_0} y \\
x\langle s\rangle_{\varphi_0}^2 y &= x \langle s \rangle_{\varphi_0} x \langle s \rangle_{\varphi_0} y \\
&\vdots \\
x\langle s\rangle_{\varphi_0}^n y &= x \langle s \rangle_{\varphi_0} x\langle s\rangle_{\varphi_0}^{n-1} y \\
&\vdots
\end{aligned}$$

We wish to show that:

$$\alpha_{\varphi_0}\left(s, x\langle s\rangle_{\varphi_0}^n y\right) = \alpha_{\varphi_0}(s,y) + n$$

We will do so by a difficult proof by induction, it is true for the first value, we are trying to show that since it is true for the first, and true for arbitrarily large values, it then holds for the values in between 1 and there. But doing so is tricky. To begin we know that:

$$\alpha_{\varphi_0}(s,y) + 2 = \alpha_{\varphi_0}(s, x\langle s\rangle_{\varphi_0} y) + 1$$

So we are asking that:

$$\alpha_{\varphi_0}\left(s, x\langle s\rangle_{\varphi_0}^2 y\right) = \alpha_{\varphi_0}(s, x\langle s\rangle_{\varphi_0} y) + 1$$

Well, we know there exists some φ_1, by Theorem 1.8.1, such:

$$\alpha_{\varphi_1}\left(s, x\langle s\rangle_{\varphi_1}^2 y\right) = \alpha_{\varphi_1}(s, x\langle s\rangle_{\varphi_1} y) + 1$$

Additionally we know there exists some value φ_1' such that:

$$\alpha_{\varphi_1'}\left(s, x\langle s\rangle_{\varphi_1'}^2 y\right) = \alpha_{\varphi_1'}(s,y) + 2$$

The goal is to show all three of these values are equal. This follows pretty plainly through analytic continuation. We now ask that we move all three of these solutions as $s \to 0$. Then each of these solutions converge towards φ_0.

The function φ_0 solves the *unique* implicit solution at $s = 0$ to all three of the above equations. This is redundant.

At this point in the proof, the identification that $\varphi_0 = \varphi(s, y) = \varphi_1 = \varphi(s, x \langle s \rangle_{\varphi(s,y)} y)$, becomes the central focus. And we close argument. $\square$

This leads us into the harder part, which is asking if this zero set continues further as s leaves the neighborhood of 0. This becomes the central focus of this endeavour. Which lies in the complex plane.

Chapter 2

The modified Bennet operators as objects in the complex plane

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