How Bennet Becomes Goodstein

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Abstract

Using Bennet's commutative operations, we construct a holomorphic Goodstein operation. This means, we construct a function $\alpha \langle s \rangle y$; for $y \in \mathbb{C}/\{0\}$ and $0 \leq \Re(s) \leq 2$; such that $\alpha \langle s \rangle (\alpha \langle s+1 \rangle y) = \alpha \langle s+1 \rangle (y+1)$. Where additionally we have that $\alpha \langle 0 \rangle y = \alpha + y$, $\alpha \langle 1 \rangle y = \alpha \cdot y$, and $\alpha \langle 2 \rangle y = \alpha^y$. This paper is written with an expectation that the reader has a graduate level of understanding of Complex Analysis and Complex Dynamics. Specifically, one should have a familiarity with the dynamics of $\exp_b(z) = b^z$.

1 Introduction

Bennet's commutative hyper-operators are typically, what one would call a curiousity. They help solve problems in iteration theory; but they're a sort of half way point between the things you actually want to study in Dynamics. As this paper is trying to be self contained–we are going to draw out every thread. So, to begin; Bennet's operations can be defined.

$$
\begin{aligned}
\alpha \oplus_{n,b} z &= \exp_b^{\circ n}\left(\log_b^{\circ n}(\alpha) + \log_b^{\circ n}(z)\right)\\
\gamma \oplus_{n+1,b} (\alpha \oplus_{n,b} z) &= (\gamma \oplus_{n+1,b} \alpha) \oplus_{n,b} (\gamma \oplus_{n+1,b} z)\\
\alpha \oplus_{n,b} z &= z \oplus_{n,b} \alpha\\
\gamma \oplus_{n,b} (\alpha \oplus_{n,b} z) &= (\gamma \oplus_{n,b} \alpha) \oplus_{n,b} z\\
\alpha \oplus_{0,b} z &= \alpha + z\\
\alpha \oplus_{1,b} z &= \alpha \cdot z\\
\alpha \oplus_{2,b} z &= \alpha^{\log(z)}
\end{aligned}
$$

These form, what is called a commutative chain of operations. We are going to ignore the domains that α and z belong to momentarily, just to give the idea of what's going on. But for $n \in \mathbb{Z}$, this produces a chain of equations.

Bennet's operations are great and all; but they're not going to get us anywhere unless we make sense of $\oplus_{s,b}$ for $s \in \mathbb{C}$. And we can't get there without fractional iterates of exponentials/logarithms. A good amount of this paper will be focused on $\exp^{\circ s}(x)$ and $\log^{\circ s}(y)$; and how we can call these things meaningful.

We can begin by defining $\oplus_{s,b}$; which comes from simply substituting $n \in \mathbb{Z} \mapsto s \in \mathbb{C}$. This becomes:

$$\begin{aligned}
\alpha \oplus_{s,b} z &= \exp_b^{\circ s}\left(\log_b^{\circ s}(\alpha) + \log_b^{\circ s}(z)\right)\\
\gamma \oplus_{s+1,b} (\alpha \oplus_{s,b} z) &= (\gamma \oplus_{s+1,b} \alpha) \oplus_{s,b} (\gamma \oplus_{s+1,b} z)\\
\alpha \oplus_{s,b} z &= z \oplus_{s,b} \alpha\\
\gamma \oplus_{s,b} (\alpha \oplus_{s,b} z) &= (\gamma \oplus_{s,b} \alpha) \oplus_{s,b} z
\end{aligned}$$

We are not going to talk about domains yet; and where these objects are holomorphic. But they are holomorphic locally everywhere. At worst we will get branch cuts; except in b, where we are restricted to a specific domain. That domain being the Shell-Thron region. As to that, we introduce how we're going to talk about $\exp^{\circ s}$. This requires defining a set and a function which will appear throughout this paper.

We define the Shell-Thron region, as:

$$\mathfrak{S} = \{b \in \mathbb{C} \mid \left|\lim_{n\to\infty} \exp_b^{\circ n}(0)\right| < \infty\}$$

This amounts to an observation, which is more the style of this paper. Let $y \in \mathbb{C}/(-\infty, 0]$, and let's let:

$$b = y^{1/y}$$

Then the function $\exp_b(z)$ has at least one fixed point z_0 such that $\exp_b(z_0) = z_0$ and $0 \leq |\log(b)z_0| \leq 1$. This means for each of these b, we have an attracting, or neutral, fixed point. This means we can iterate $\exp_b(x)$ on very explicit domains, but, we can do this broadly. This relates to the Shell-Thron region, in that, $b \in \mathfrak{S}$ if $b = y^{1/y}$. Tous que ca dit, c'est: we're only going to talk about iterated exponentials for $b \in \mathfrak{S}$–luckily this is all we'll need.

Iterated exponentials in the Shell-Thron region are well studied and well understood. We point to Kouznetsov and Trappman, for the ideal form of these iterated exponentials. This produces what is called regular iteration, or Schröder's iteration of the exponential. The last step we need is, there exists multiple iterated exponentials within the Shell-Thron region. This is delineated by the β method of iterated exponentials; but we won't need much from the β method. As to that, we begin this paper examining how we can iterate exponentials.

2 How do we iterate $\exp_b$?

The manner we can begin this conversation, is twofold. We can talk about β up front, or we can talk about regular iteration and derive β. This means, we can talk about the varying types of iterates of $\exp_b$; or we can talk about the "best" iterate of $\exp_b$. We will choose the latter, as the intention is to be slow and thorough.

To choose the latter, means it's beneficial to start even earlier. And by that, we'll begin by arguing by example. Let's let $b = 2^{1/2} = \sqrt{2}$. Let's describe what we want to solve:

$$\begin{aligned} \sqrt{2}^{F(s)} &= F(s+1) \\ F(0) &= 1 \end{aligned}$$

To begin, the function $f(z) = \sqrt{2}^z$ has an attracting fixed point at $z = 2$. Such that, $f(2) = 2$ and $f'(2) = \log(2)$. In a neighborhood of 2, we have a Schröder function ψ, such that:

$$\psi(f(z)) = \log(2)\psi(z)$$

This can be inverted, to give:

$$\exp_{\sqrt{2}}^{\circ s}(z) = \psi^{-1}(\log(2)^s(\psi(z)))$$

This works for $s \in \mathbb{C}$ (upto branch cuts), and $z \in \mathcal{A}$, where $\mathcal{A}$ is the immediate basin of attraction of 2. So essentially, this works everywhere:

$$\lim_{n\to\infty} \exp^{\circ n}(z) = 2$$

But only for the maximally connected domain which includes 2. This is what's known as Schröder iteration. To get regular iteration from Schröder iteration, we have to expand our domain a bit. The key to regular iteration is to consider the fixed point at 4 as well.

We know that:

$$f(4) = 4 = \sqrt{2}^4$$

Which is a repelling fixed point, $f'(4) = 2\log(2)$. We can perform Schröder iteration here, but now we'd talk about $f^{-1}(z) = \log_{\sqrt{2}}(z)$ in a neighborhood of $z = 4$. Pasting these solutions together, produces what we refer to as regular iteration. When I write:

$$\exp_{\sqrt{2}}^{\circ s}(z)$$

We will assume that we are choosing regular iteration, which works for repelling or attracting fixed points. We will only ever care about z which is near-ish a fixed point; so a lot of the trouble will take care of itself.

Now, within the Shell-Thron region everything behaves identically to $\sqrt{2}$. There are little differences. The trouble which arises with regular iteration, is when we hit a neutral fixed point. This occurs exactly on the boundary of the Shell-Thron region.

Again, we will argue by example, and choose Trappman's constant $\eta = e^{1/e}$. This constant dates to Euler, but as Euler has already done everything any mathematician will ever do; the consensus is to call this Trappman's η constant. This value is important because:

$$\begin{aligned} h &= \eta^z \\ h(e) &= e \\ h'(e) &= 1 \end{aligned}$$

We can no longer perform a Schröder iteration. We can still perform regular iteration, though. It will behave nearly identical, but we'll have more volatile behaviour. This means we will have two very distinct iterations: η^+, η^-. Here, η^- iterates the same for the Fatou set; but iterates poorly for the Julia set; whereas, η^+ iterates the Julia set perfectly. We have a discontinuity if we try to rectify these solutions.

Nonetheless, we obtain two iterations such that:

$$\begin{aligned} &\eta^\pm(s+1) = \eta^{\eta^\pm(s)} \\ &\lim_{s\to\infty} \eta^-(s) \to e \\ &\lim_{s\to\infty} \eta^+(s) \to \infty \end{aligned}$$

We will ignore branch-cuts momentarily, and the domains of holomorphy; but this is the central idea. This covers everywhere on the boundary of the Shell-Thron region. This means, if I write $b = y^{1/y}$ for $y \in \mathbb{C}/\{0\}$–then we can write an iterate $\exp_b^{\circ s}(z)$, almost everywhere in z and almost everywhere in s. This is what will be referred to as regular iteration. As the goal is always to consider $z \approx y$ where $b^y = y$; we don't need to worry too much about technicalities. We can iterate near attracting, repelling, or neutral fixed points.

3 So what's the β method?

The β method is a quite intricate construction. But for us, it means a single thing. It does just one thing. That is, it controls the period of our iteration.

This means, it lets us have control of the period of the iteration. To return to $b = \sqrt{2}$; the explanation is simple. Using, regular iteration, we can construct:

$$\sqrt{2}^{F(s)} = F(s+1)$$

But since we've defined this as:

$$F(s) = \exp_{\sqrt{2}}^{\circ s}(z) = \psi^{-1}(\log(2)^s(\psi(z)))$$

We must have that:

$$\exp_{\sqrt{2}}^{\circ s + 2\pi i/|\log\log(2)|} = \exp_{\sqrt{2}}^{\circ s}$$

The β method essentially lets us choose the period. This means, we introduce a new parameter λ; where $\Re\lambda > 0$; such:

$$F(s + 2\pi i/\lambda) = F(s)$$

This means, we can create more than one iteration of $\exp_{\sqrt{2}}(z)$. We can create many, and they are determined by what we choose for our period. So, if I introduce the parameter λ to our notation:

$$\begin{aligned}
F_\lambda(s) &= \exp_{\sqrt{2},\lambda}^{\circ s}(z) \\
F_\lambda(s+1) &= \sqrt{2}^{F_\lambda(s)} \\
F_\lambda(s + 2\pi i/\lambda) &= F_\lambda(s) \\
F_{|\log\log(2)|}(s) &= \exp_{\sqrt{2}}^{\circ s}(z) \text{ is the regular iteration}
\end{aligned}$$

The β method gives us a manner of constructing these things for all $\Re(\lambda) > |\log\log(2)|$. The same thing occurs throughout the interior of the Shell Thron region. On the boundary we have an even better result, but it may be difficult to convey. Again, Trappman's constant can help us visualize it. The function $\eta^\pm$ does not have a period; as do all values on the border of the Shell Thron region–they have neutral fixed points. The β method allows us to add a period. This means we have an $\eta^\pm_\lambda$ such that:

$$\begin{aligned}
\eta^\pm_\lambda(s+1) &= \eta^{\eta^\pm_\lambda(s)} \\
\eta^\pm_\lambda(s + 2\pi i/\lambda) &= \eta^\pm_\lambda(s)
\end{aligned}$$

This occurs everywhere on the boundary of the Shell-Thron region $\mathfrak{S}$. The sole difference to the interior, is the β method works for all $\Re\lambda > 0$ on the boundary. Normally the β method only works for $\Re\lambda > |\log(y \log b)|$. This means we have a family of holomorphic functions:

$$
\begin{aligned}
b &= y^{1/y}\\
q_{b,\lambda}(s,z) &= \exp_{b,\lambda}^{\circ s}(z)\\
|z-y| &< \delta\\
q_{b,\lambda}(s+1,z) &= \exp_b(q_{b,\lambda}(s,z)) = b^{q_{b,\lambda}(s,z)}\\
q_{b,\lambda}(s+2\pi i/\lambda,z) &= q_{b,\lambda}(s,z)
\end{aligned}
$$

This is all the β method means. We sacrifice a lot of domains in this construction. But, it's always holomorphic almost everywhere for $s,z \in \mathbb{C}$.

4 Setting up the implicit function theorem

We can almost write our desired expression, but to get there we have to be careful. The major result of this paper is to use the implicit function theorem; and to do that, we have to be careful. To begin, we write our potential solution:

$$G_\lambda(\alpha,y,s) = \alpha \oplus_{s,y^{1/y}} y$$

This function, looks precisely like this:

$$G_\lambda(\alpha,y,s) = \exp_{y^{1/y},\lambda}^{\circ s}\left(\log_{y^{1/y},\lambda}^{\circ s}(\alpha) + y\right)$$

Regardless of λ, this function perfectly weaves between addition, multiplication and exponentiation.

$$
\begin{aligned}
G_\lambda(\alpha,y,0) &= \alpha + y\\
G_\lambda(\alpha,y,1) &= \alpha \cdot y\\
G_\lambda(\alpha,y,2) &= \alpha^y
\end{aligned}
$$

Now, the functional equation we want to solve is Goodstein's equation which equates to:

$$G_\lambda(\alpha, G_\lambda(\alpha,y,s), s-1) = G_\lambda(\alpha, y+1, s)$$

This is always solved for $s = 0,1,2$, as it just looks like:

$$
\begin{aligned}
\alpha + \alpha\cdot y &= \alpha\cdot(y+1)\\
\alpha\cdot\alpha^y &= \alpha^{y+1}
\end{aligned}
$$

So we just want to make an implicit function in λ, such that this continues to hold as we move s. This is accomplished using the implicit function theorem, and a monodromy theorem. The only real difficult part is showing this can be validly solved.

5 Non-zero derivative of iterated exponentials.

We can begin by looking at $b = \sqrt{2}$ again, or $y = 2$. We want to show that we have a non-zero derivative for this iterate. To begin, let:

$$F(s,z) = \exp_{\sqrt{2}}^{\circ s}(z)$$

Then,

$$\frac{\partial}{\partial s}F(s+1,z) = \log(2)F(s+1,z)\frac{\partial}{\partial s}F(s,z)/2$$

Therefore the function $\frac{\partial}{\partial s}F(s,z) \neq 0$ unless $F(s) = 0$, but if $F(s_0) = 0$ then $F(s_0 - 1)$ is a logarithmic singularity, and thus $F(s_0) \cdot \frac{\partial}{\partial s}F(s_0 - 1, z) \neq 0$.

The variable in z is a little trickier. But, there exists a function h, such that:

$$F(s,z) = \exp_{\sqrt{2}}^{\circ s+h(z)}(a)$$

Where h is the functional inverse of $F(s,a)$; and therefore non-zero. Therefore the derivative in z of $F(s,z)$ is non zero. Further, we get that both partial derivatives are zero, or that:

$$\frac{\partial^2 F(s,z)}{\partial s \partial z} \neq 0$$

Since these are non-singular values, we will never run into any problems, when trying to make an implicit function in λ. To clarify, the value λ, produces a value θ, such that:

$$\exp_{b,\lambda}^{\circ s}(z) = \exp_b^{\circ s+\theta}(z)$$

Which is always non-singular, and thus we can talk freely about the implicit function theorem now.

6 The implicit function theorem locally

Momentarily we are going to consider $s \approx 1, 2$, and look at our desired formula– and note what we have wrong so far. Let us write:

$$g(\alpha, y, s, \lambda) = G_\lambda(\alpha, G_\lambda(\alpha, y, s), s-1) - G_\lambda(\alpha, y+1, s)$$

And we want to solve for λ such that:

$$g(\alpha, y, s, \lambda(\alpha, y, s)) = 0$$

Therefore we need to show the Jacobian is non-zero; which is not so simple for us, because it's not true. We can actually knock this out pretty quickly, but we will hold off for a moment. The more difficult part is showing this extends beyond the local scenario, which requires a use of the monodromy theorem. For the moment, we just care about $s \approx 1, 2$. We will handle the Jacobian only for the more difficult case of the monodromy theorem.

It's helpful to note that for $y = 1$ that these equations produce a degeneracy, so this method works precisely for $y \neq 0, 1$–which luckily happen to be the identity elements of the operators we are interpolating, which is very helpful. The value α works almost everywhere, excusing where $\log^{\circ s}_{y^{1/y},\lambda}(\alpha) = \infty$ and where $\exp^{\circ s}_{y^{1/y},\lambda}\left(\log^{\circ s}_{y^{1/y},\lambda}(\alpha) + y\right) = \infty$.

The big trouble, is the following sequence of identities:

$$\begin{aligned} g(\alpha, y, 1, \lambda) &= 0 \\ g(\alpha, y, 2, \lambda) &= 0 \\ \frac{\partial}{\partial \lambda} g(\alpha, y, 1, \lambda) &= 0 \\ \frac{\partial}{\partial \lambda} g(\alpha, y, 2, \lambda) &= 0 \end{aligned}$$

Which guarantees the implicit function theorem does not have a unique solution. We actually get a branching problem near these two points. The problem at hand is that we know now that there's either multiple solutions, or *zero solutions*. We want to show that we get multiple solutions, and not zero solutions. What is guaranteed though, is that in a neighborhood of $s \approx 1, 2$, that we still have the existence of these points which satisfy the equation.

This comes from defining the function $s(\lambda)$ instead of $\lambda(s)$, near the points $s = 1, 2$. We will have multiple solutions to this equation, as we can choose $s(\lambda) = 1, 2$ for any λ. Once we make this choice, we will have a function $\lambda(s)$ for $1 < s < 2$, with an unknown type of singularity here. It is most definitely a type of branch cut.

This provides us with a more difficult problem, and incites us to introduce a more nuanced approach. Thus, enters a more delicate parameter.

7 The φ parameter

Let us take all that we know, and let's change it up a little bit. Let's define a new function, which plays in the variable of y. Let's write:

$$\exp^{\circ s}_{b,\lambda}(\log^{\circ s}_{b,\lambda}(\alpha) + y) = \exp^{\circ s}_{b}(\log^{\circ s}_{b}(\alpha) + \varphi + y)$$

The function φ is holomorphic where-ever the left hand side is holomorphic. We are going to talk, instead of an implicit function in λ, as an implicit function in φ–which then, $\lambda = \lambda(\varphi)$, which will have a branching problem at $s = 1, 2$; but φ won't have this problem.

This nearly solves the above problem. But now, we're going to consider $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \lambda_3)$; and its equivalent $\boldsymbol{\varphi} = (\varphi_1, \varphi_2, \varphi_3)$. This means, as above we considered the use of λ as 1-dimensional, we will now consider three variables.

This means, we are looking at:

$$g(\alpha, y, s, \boldsymbol{\lambda}) = G_{\lambda_1}(\alpha, G_{\lambda_2}(\alpha, y, s), s-1) - G_{\lambda_3}(\alpha, y+1, s)$$

And now, implicitly each of these values always exist, and furthermore are non-degenerate:

$$\frac{\partial}{\partial \lambda_j} g \neq 0$$

This can be seen by just considering $G_\lambda(\alpha, y, s) = G(\alpha, y + \varphi, s)$; which is always non-degenerate. This solution will continue beyond, and allow for the monodromy theorem; as we can move each λ freely.

The difficulty now, is to identify that, each λ is a function of different variables. This becomes the real tricky part. We want λ_1 to be a function of $\alpha, G_{\lambda_2}, s-1$; λ_2 to be a function of α, y, s; and λ_3 to be a function of $\alpha, y+1, s$. Such that, we can always reconcile this into one $\lambda(\alpha, y, s)$–which is simple to do. It just requires a nice change of variables. This means, we add the additional constraint:

$$\lambda_1(\alpha, y, s) = \lambda_2(\alpha, y, s) = \lambda_3(\alpha, y, s)$$

Once we make the change of variables. This ensures we do not get the degenerate problem as above, as we freely move each occurrence of λ, and λ becomes one function. This then becomes one function φ. Returning to Bennet's commutative operations, we obtain our Goodstein operators:

$$\alpha \langle s \rangle y = \alpha \oplus_{s, y^{1/y}} (y + \varphi(\alpha, y, s))$$

This solution can be found for all $0 \leq \Re(s) \leq 2$, as an implicit solution can always be found for these values–subtracting $y = 0, 1$, and ignoring some singular values of α for varying s, y. The monodromy theorem takes care of everything at this point.

We always have a local solution in φ, which accounts for 3 neighborhoods $(\varphi_1, \varphi_2, \varphi_3)$, such that:

$$\alpha \langle s-1 \rangle (\alpha \langle s \rangle y) = \alpha \langle s \rangle (y+1)$$

Where we are guaranteed that $\varphi(\alpha, y, 0), \varphi(\alpha, y, 1), \varphi(\alpha, y, 2) = 0$.

We've played pretty fast and loose throughout this paper. All of that is about to change. We are going to work exclusively with φ. And we are only going to work exclusively with regular iteration. We are going to drop λ from the discussion, and just as similarly, we will drop the β method. It has its relation to this study. But to rigorously set up the implicit function theorem, it is useless.

8 Off into the deepend...

We start by taking an intermediary operator. For $\varphi \in \mathbb{C}$, we define:

$$\alpha \langle s \rangle_{\varphi} y = \exp^{\circ s}_{y^{1/y}} \left(\log^{\circ s}_{y^{1/y}}(\alpha) + y + \varphi \right)$$

There exists a surface of values, for $\boldsymbol{\varphi} \in \mathbb{C}^3$, such that:

$$\alpha \langle s-1 \rangle_{\varphi_1} \alpha \langle s \rangle_{\varphi_2} y = \alpha \langle s \rangle_{\varphi_3} y + 1$$

Such when $s \approx 1, 2$, we have $\boldsymbol{\varphi} \approx (0,0,0)$. As we move $(\alpha, y, s) \in \mathbb{C}^3$; we move the solutions $\boldsymbol{\varphi}$; but keeping $(\alpha, y, s) \approx (\alpha, y, 1)$, $(\alpha, y, 2)$–we are guaranteed $\boldsymbol{\varphi} \approx \mathbf{0}$. Because of these parameters, we can create a surface $\boldsymbol{\Phi} \subset \mathbb{C}^3$.

This allows us to choose a path on this surface. In which, we're solving:

$$\begin{aligned} \Phi \ni \boldsymbol{\varphi}(\alpha, y, s) &: \mathbb{C}^3 \to \mathbb{C}^3 \\ (\alpha, y, s) &\mapsto \boldsymbol{\varphi} = (\varphi_1, \varphi_2, \varphi_3) \end{aligned}$$

We can take ϵ changes in (α, y, s) and affect φ. We are to now, not look at $\boldsymbol{\varphi} : \mathbb{C}^3 \to \mathbb{C}^3$; and instead look at $\varphi : \mathbb{C}^3 \to \mathbb{C}$. Given the values already solved for, we are choosing a path along $\boldsymbol{\Phi}$, which creates a holomorphic function in (α, y, s). This is epitomized by:

$$\begin{aligned} \varphi(\alpha, \alpha \langle s \rangle_{\varphi(\alpha,y,s)} y, s-1) &= \varphi_1(\alpha, y, s) \\ \varphi(\alpha, y, s) &= \varphi_2(\alpha, y, s) \\ \varphi(\alpha, y+1, s) &= \varphi_3(\alpha, y, s) \end{aligned}$$

This system of equations always exists, and is always solvable. This gives us our solution. But it gives us no hope of constructing the solution, for that we'll have to be more clever. But forgoing this, an implicit solution always exists to:

$$\begin{aligned} \alpha \langle s \rangle_{\varphi(\alpha,y,s)} y &= \alpha \langle s \rangle y \\ \alpha \langle s-1 \rangle (\alpha \langle s \rangle y) &= \alpha \langle s \rangle (y+1) \end{aligned}$$

To get the proper solution in a constructive manner, we'll have to introduce differential equations. This will produce a set of difficult differential equations, which are to say the least, rather thick and wordy.

9 Including differential theory

To construct the solution φ properly, it is important to add differential theory. This gives us a picture of what the implicit solution will look like. And does so as we move the variables along the surface $\mathbf{\Phi}$. We need to think slightly different now.

The surface $\mathbf{\Phi} \subset \mathbb{C}^3$ will play a more prominent role here. We want to, in no small terms, trace a path on this surface. We begin by writing, $\boldsymbol{\varphi}(\alpha, y, s) = (\varphi_1, \varphi_2, \varphi_3)$. We can define the operation at hand:

$$\begin{aligned} g(\alpha, y, s, \boldsymbol{\varphi}) &= \alpha \langle s-1 \rangle_{\varphi_1} (\alpha \langle s \rangle_{\varphi_2} y) - \alpha \langle s \rangle_{\varphi_3} (y+1) \\ &= 0 \end{aligned}$$

We don't need to talk about α at all, and can consider it a constant. So we can write $g(y, s, \boldsymbol{\varphi})$ without confusion. We can start by differentiating in y. We will write $\alpha \langle s \rangle'_{\varphi} y = \frac{\partial}{\partial y} \alpha \langle s \rangle y$, when we hold φ constant. With that, we get:

$$\begin{aligned} \frac{d}{dy} \alpha \langle s \rangle_{\varphi_3} y + 1 &= \big(\alpha \langle s \rangle'_{\varphi_3} y + 1\big) + \frac{d\varphi_3}{dy} \frac{\partial}{\partial \varphi_3} \alpha \langle s \rangle_{\varphi_3} (y+1) \\ &= \alpha \langle s-1 \rangle'_{\varphi_1} (\alpha \langle s \rangle_{\varphi_2} y) \left(\alpha \langle s \rangle'_{\varphi_2} y + \frac{d\varphi_2}{dy} \frac{\partial}{\partial \varphi_2} \alpha \langle s \rangle_{\varphi_2} y \right) + \frac{d\varphi_1}{dy} \frac{\partial}{\partial \varphi_1} \alpha \langle s-1 \rangle_{\varphi_1} (\alpha \langle s \rangle_{\varphi_2} y) \end{aligned}$$

Now subbing in the appropriate appropriate variable changes, we have a first order delay differential equation in φ in the variable y. This effectively sets up an Eulerian method of construction, and where we'll end.