A Tetration Function By Unconventional Means

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Abstract

The author makes use of infinite compositions and a limiting function to sketch the construction of a holomorphic tetration function $\mathcal{F}(s) = e \uparrow\uparrow s$. As a tetration function, $\mathcal{F}$ satisfies $e^{\mathcal{F}(s)} = \mathcal{F}(s+1)$. Of it, $\mathcal{F}$ is holomorphic on the domain $\mathbb{C}/\mathcal{L}$ where $\mathcal{L}$ is a nowhere dense set; and $\mathcal{F}$ takes $(-2,\infty) \to \mathbb{R}$ bijectively with strictly monotone growth.

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1 Introduction

The study of tetration can be traced back centuries. The first notable moment being, when Leonhard Euler proved that when $e^{-e} < \alpha < e^{1/e}$; the infinite tower converges [2]. Quote unquote, for such α,

$$\alpha^{\alpha^{\cdots (n\,\text{times}) \cdots^{\alpha}}} \to A \text{ as } n \to \infty$$

Where $\alpha^A = A$ and $e^{-1} \leq A \leq e$. This was a monumentous moment in the study of iterated exponentials; and is probably the first truly publishable result in the history of tetration–hell, complex dynamics. Though the subject of tetration remains rather dormant; if we do not count numerous forays into the complex dynamics of the exponential function; it poses itself as a very interesting problem. The author would like to think, at the center of the study of iterated exponentials is the study of tetration functions; and the quest for a good and right solution should be a priority. But as the pulse of the subject is weak, he feels not many others feel the same way.

A tetration function is simple enough to describe, but proves a very difficult function to construct (if we want a good and right solution). We'll restrict our attention to tetration functions with the base e. For bases $b > e^{1/e}$ the result is similar. If $\mathcal{F}$ is a function such that $e^{\mathcal{F}(s)} = \mathcal{F}(s+1)$ and $\mathcal{F}(0) = 1$, then we call $\mathcal{F}$ a tetration function. It provides a similar continuation of the sequence

$e, e^e, e^{e^e}, ...$ as e^s provides a continuation of the sequence $e, ee, eee, ...$. In contrast to the exponential function though, it is a much more volatile construction.

One facet of its volatility, tetration functions are highly non-unique. For any tetration function $\mathcal{F}$ the function $\mathcal{F}(s+\sin(2\pi s))$ is also a tetration function; as is $\mathcal{F}(s+\theta(s))$ for any 1-periodic function. Alors, a large problem with tetration, is qualifying a tetration function as unique, or as satisfying some property which characterizes it from other tetration functions.

There are quite a few trivial ways to construct a tetration function, for instance letting $\mathcal{F}(t) = 1 + t$ for $t \in [-1, 0]$ and extending $\mathcal{F}$ to $\mathbb{R}^+$ using the functional equation provides such a solution. But necessarily such a solution is not differentiable on the natural numbers. One can construct many continuous extensions in a parallel manner.

If we were to ask for a tetration function, we would at least require that it be analytic. Even better, that it be holomorphic on some domain in the complex plane including $\mathbb{R}^+$.

Of that end, Hellmuth Kneser was the first to undoubtedly provide a desirable solution to the tetration equation [3]. Insofar as it took the real positive line to itself and was holomorphic. He worked exclusively with the base $b = e$, and managed to construct a function $\mathcal{H}(s) = {}^s e$ holomorphic in $\mathbb{C}$ excluding a nowhere dense set. His construction, although providing a stable solution, was highly esoteric and deeply expressed the difficulty of this problem. Constructing a holomorphic tetration function is no easy feat. In his defense, his sole goal was to construct h such that $h(h(z)) = e^z$ and h was real-valued.

More recently, numerous attempts have been made to construct a simpler tetration function. There exists quite a few potential candidates for tetration; which exist scattered in the recesses of the internet. Since tetration has not gained widespread recognition as a notable problem–it is difficult to find published papers on the subject. Kneser's own paper [3] is in German, and there exists no English translations–but there are synopses and breakdowns.

The problem with most modern approaches to tetration seem to be that these candidates can be numerically verified, but never rigorously justified to exist or converge. And in contrast; Kneser's construction, which is correct, is a rather laborious numerical procedure. Even more troubling with these flurry of candidates to tetration; proofs of analycity become even more difficult–and in most instances do not exist. Alongside a lack of proof for mere convergence, this can be troubling. If that wasn't bad enough; we can't even prove if two candidate tetration functions equal or not. But sometimes our floating point accuracy can appear to suggest so (or dissuade so).

Nonetheless there is still great headway being made in the field. A treasured example is in the work of Dmitri Kouznetsov [4]. The author will not go into detail on Kouznetsov's method but will simply give a rough heuristic as motivation. The author will blatantly steal this heuristic–but he shall bypass a few of the obstacles he believes Kouznetsov built for himself. Though, in truth, much of the work parallels an extension of his trick.

Supposing we had a nice function $g(s)$, which has some desirable growth properties, and we took the limiting function,

$$G_n(s) = \log\log\cdots(n\,\text{times})\cdots\log g(s+n)$$

Then $e^{G_n(s)} = G_{n-1}(s+1)$. If the limit were to converge as $n \to \infty$, then we would have our tetration function $G_n \to G$. That is, upto a normalization constant ω where $\mathcal{F}(s) = G(s+\omega)$ to ensure $\mathcal{F}(0) = 1$.

Kouznetsov chose a very wonderful function g, such that numerically everything worked out and provides us with a calculator's version of a holomorphic tetration function. The function g was constructed through careful fixed-point analysis; and looks like an exponential sum of terms e^{nLs} for L a complex fixed point of e^s. Unfortunately, to rigorously justify convergence proves rather difficult. This becomes a sort of brick-wall. There is no in your hands proof that Kouznetsov's method actually works.

To that end, the goal of this paper is to construct our own g, and show the convergence of the above limit. Our choice of g will be very manufactured, and requires a familiarity with infinite compositions. To that end, we refer the reader to [6], where sufficient conditions are provided for an infinite composition to converge–and a familiarity with the subject is created. We shall not need anything from [6], but it exhibits the nuanced detail of the subject a bit more clearly. We will prove a modified form of the main result of [6]; to keep this paper self-contained; but we will give little to no motivating intuition in this paper.

The essential trick is to bypass the difficulty of constructing a tetration function using Kouznetsov's trick by constructing a function which satisfies a similar functional equation, but exhibits the same growth properties. As you may guess, tetration grows rapidly and rabidly in the complex plane, so we'll need something similarly chaotic.

To that end, our first goal is to construct an entire function $\phi(s)$ such that,

$$\phi(s+1) = e^{s+\phi(s)}$$

And take ϕ to be our g from above. The real novelty of the work is held in constructing ϕ. But it only really takes us writing out the equation for ϕ and justifying convergence. This is more of a taxing process than a difficult one. It is surprisingly simple to construct ϕ.

* * *

We introduce briefly the notation for nested compositions, which allows for our construction of ϕ. We will restrict from full generality, and only care about a subset of types of infinite compositions. Therein, if $h_j(s,z)$ is a sequence of entire functions in both variables, then,

$$\underset{j=1}{\overset{n}{\Omega}} h_j(s,z) \bullet z = h_1(s, h_2(s, ...h_n(s,z)))$$

Where we are interested in letting $n \to \infty$. The study of infinite compositions is very nuanced, and for that reason constructing ϕ will require care. The type of convergence we'll need is one which is a bit simpler than the general case. To wit, we will call,

$$\phi_n(s) = \Omega_{j=1}^{n} e^{s-j+z} \bullet z\Big|_{z=0}$$

Where if $h_j(s,z) = e^{s-j+z}$ then,

$$\phi_n(s) = h_1(s, h_2(s, ...h_n(s, 0)))$$

By design, $e^{s+\phi_n(s)} = \phi_{n+1}(s+1)$, so if this were to converge it would equal our desired function. The essential ingredient in our construction is for all compact disks $\mathcal{P}, \mathcal{K} \subset \mathbb{C}$,

$$\sum_{j=1}^{\infty} ||h_j(s,z)||_{s\in\mathcal{P},z\in\mathcal{K}} < \infty$$

Where this is taken to mean the supremum norm. So without further ado, we construct ϕ.

2 Constructing ϕ

The first thing we need to construct ϕ is a sort of normality condition. For all $\epsilon > 0$, there exists some N, such when $m \geq n > N$,

$$|\Omega_{j=n}^{m} e^{s-j+z} \bullet z| < \epsilon$$

For $|z| < 1$, and s residing in some compact disk within $\mathbb{C}$. This then implies as we let $m \to \infty$, the tail of the infinite composition stays bounded. Forthwith, the infinite composition becomes a normal family, and proving convergence becomes simpler. We provide a quick proof of this.

Lemma 2.1. *For a compact disk $\mathcal{P} \subset \mathbb{C}$ and $|z| \leq 1$: for all $\epsilon > 0$, there exists some N, such when $m \geq n > N$*

$$||\Omega_{j=n}^{m} e^{s-j+z} \bullet z||_{\mathcal{P},|z|\leq 1} < \epsilon$$

Proof. Let $|z| \leq 1$ and $s \in \mathcal{P}$ be a compact disk in $\mathbb{C}$. Set $h_j(s,z) = e^{s-j+z}$ and set $||h_j(s,z)||_{s\in\mathcal{P},|z|\leq 1} = \rho_j$. Pick $\epsilon > 0$, and choose N large enough so when $n > N$,

$$\rho_n < \epsilon$$

Denote: $\phi_{nm}(s,z) = \Omega_{j=n}^{m} h_j(s,z) \bullet z = h_n(s, h_{n+1}(s, ...h_m(s,z)))$. We go by induction on the difference $m - n = k$. When $k = 0$ then

$$||\phi_{nn}(s,z)||_{|z|<1,s\in\mathcal{P}} = ||h_n(s,z)||_{|z|<1,s\in\mathcal{P}} = \rho_n < \epsilon$$

Assume the result holds for $m - n < k$, we show it holds for $m - n = k$. Observe,

$$\begin{aligned}
||\phi_{nm}(s,z)||_{|z|<1,s\in\mathcal{P}} &= ||h_n(s,\phi_{(n+1)m}(s,z))||_{|z|<1,s\in\mathcal{P}}\\
&\leq ||h_n(s,z)||_{|z|<1,s\in\mathcal{P}}\\
&= \rho_n < \epsilon
\end{aligned}$$

Which follows by the induction hypothesis because $|\phi_{(n+1)m}(s,z)| < \epsilon < 1$. □

The next step is to observe that $\Omega_{j=1}^m h_j(s,z)$ is a normal family as $m \to \infty$, for $|z| < 1$ and $s \in \mathcal{P}$, an arbitrary compact disk. This follows because the tail of this composition is bounded. Therefore we can say $||\,\Omega_{j=1}^m h_j(s,z)||_{|z|<1,s\in\mathcal{P}} < M$ for all m.

From this we can prove our infinite composition converges, and construct our entire function $\phi(s)$.

Theorem 2.2. *The expression*

$$\mathop{\Omega}_{j=1}^{\infty} e^{s-j+z} \bullet z\Big|_{z=0} = \phi(s)$$

is an entire function satisfying the identity $e^{s+\phi(s)} = \phi(s+1)$.

Proof. Since $\phi_m(s,z) = \Omega_{j=1}^m e^{s-j+z} \bullet z$ are a normal family; there is some constant $M \in \mathbb{R}^+$ such,

$$||\frac{d^k}{dz^k}\phi_m(s,z)||_{|z|<1,s\in\mathcal{P}} \leq M \cdot k!$$

Secondly, using Taylor's theorem,

$$\begin{aligned}
\phi_{m+1}(s,z) - \phi_m(s,z) &= \phi_m(s,e^{s-m-1+z}) - \phi_m(s,z)\\
&= \sum_{k=1}^{\infty}\frac{d^k}{dz^k}\phi_m(s,z)\frac{(e^{s-m-1+z}-z)^k}{k!}\\
&= (e^{s-m-1+z}-z)\sum_{k=1}^{\infty}\frac{d^k}{dz^k}\phi_m(s,z)\frac{(e^{s-m-1+z}-z)^{k-1}}{k!}
\end{aligned}$$

Setting $z = 0$, the series on the right can be bounded by some $C \in \mathbb{R}^+$. Applying the obvious bounds,

$$||\phi_{m+1}(s,0) - \phi_m(s,0)||_{s\in\mathcal{P}} \leq C||e^{s-m-1}||_{s\in\mathcal{P}} = Ae^{-m}$$

For some $A \in \mathbb{R}^+$. And we can see the telescoping series converges and $\phi_m(s)$ must be uniformly convergent for $s \in \mathcal{P}$, and therefore defines a holomorphic function $\phi(s)$ as $m \to \infty$. Naturally $e^{s+\phi_m(s)} = \phi_{m+1}(s+1)$, and so therefore the functional equation is satisfied. $\square$

3 The correction term τ

The main philosophy of our approach to constructing is to add a corrective term to ϕ such that it becomes a function. The function ϕ already looks very close to tetration, satisfying a similar functional equation,

$$\phi(s+1) = e^{s+\phi(s)}$$

We will introduce a sequence of correction terms as follows:

$$\log\log\cdots(n\,\text{times})\cdots\log\phi(s+n) = \phi(s) + \tau_n(s)$$

Where inductively, starting with $\tau_1(s) = s$ and $\tau_0(s) = 0$; τ_n can be defined,

$$\begin{aligned}
\tau_{n+1}(s) &= \log\big(\phi(s+1) + \tau_n(s+1)\big) - \phi(s)\\
&= \log\phi(s+1) + \log\big(1 + \frac{\tau_n(s+1)}{\phi(s+1)}\big) - \phi(s)\\
&= s + \phi(s) + \log\big(1 + \frac{\tau_n(s+1)}{\phi(s+1)}\big) - \phi(s)\\
&= s + \log(1 + \frac{\tau_n(s+1)}{\phi(s+1)})
\end{aligned}$$

Our choice of log is defined implicitly by the relation,

$$e^{\phi(s)+\tau_{n+1}(s)} = \phi(s+1) + \tau_n(s+1)$$

And the restriction that $\tau_n(s)$ is real on the real-line. The first thing to note, is that for $s = t \in \mathbb{R}^+$, the sequence of functions τ_n converge uniformly on bounded intervals greater than T. This is because $\tau_{n+1}(t+1)/\phi(t+1) < 1$ and using the relation $|\log(1+A) - \log(1+B)| \leq |A-B|$,

$$\begin{aligned}
|\tau_{n+1}(t) - \tau_n(t)| &\leq |\log(1 + \frac{\tau_n(t+1)}{\phi(t+1)}) - \log(1 + \frac{\tau_{n-1}(t+1)}{\phi(t+1)})| \\
&\leq \frac{1}{\phi(t+1)}|\tau_n(t+1) - \tau_{n-1}(t+1)| \\
&\leq \frac{1}{\phi(t+1)\phi(t+2)}|\tau_{n-1}(t+2) - \tau_{n-2}(t+2)| \\
&\vdots \\
&\leq \frac{1}{\prod_{k=1}^n \phi(t+k)}|\tau_1(t+n) - \tau_0(t+n)|
\end{aligned}$$

Recalling that $\tau_1(t) = t$ and $\tau_0(t) = 0$ then,

$$|\tau_{n+1}(t) - \tau_n(t)| \leq \frac{t+n}{\prod_{k=1}^n \phi(t+k)}$$

And here $\phi(t)$ is monotone increasing and unbounded, so for some T with $t > T$ it is $\frac{1}{\phi(t)} \leq \lambda < 1$. Now,

$$|\tau_m - \tau_n| \leq \sum_{j=n}^{m} \lambda^j |s + j|$$

Of sorts; the telescoping sum converges uniformly on bounded intervals and we are given a function $\tau : \mathbb{R}^+_{t>T} \to \mathbb{R}^+$. This function acts such that,

$$\widetilde{\mathcal{F}}(t) = \phi(t) + \tau(t)$$

And $\widetilde{\mathcal{F}}$ is a function, albeit not yet normalized to $\widetilde{\mathcal{F}}(0) = 1$, but there is an appropriate ω such that $\mathcal{F}(t) = \phi(t+\omega) + \tau(t+\omega)$ is a true tetration function. Also by taking logarithms, the domain can be extended to its maximal $(-2, \infty)$. Call this function $\mathcal{F}$.

Going through the same motions as above one can derive that $\mathcal{F}'$ is a continuous function and that $\tau_n' \to \tau'$ uniformly on bounded intervals. This is really no different then what we've already written. The function,

$$\tau_{n+1}'(t) = 1 + \Big(\frac{1}{1 + \frac{\tau_n(t+1)}{\phi(t+1)}}\Big)\Big(\frac{\tau_n'(t+1)}{\phi(t+1)} - \frac{\tau_n(t+1)}{\phi(t+1)^2}\phi'(t+1)\Big)$$

Grinding the gears of this expression we get $|\tau_{n+1}' - \tau_n'|$ is a convergent series of the same form as above. This is more of a task than a problem. It is left to the reader.

The next brief fact we need is that $\mathcal{F}'(t) > 0$ for all $t \in (-2, \infty)$. From the expression above, $\tau'(t) - 1 \to 0$ as $t \to \infty$. So, eventually $\tau'(t) > 0$ for some $t \geq T$. It is no hard fact to notice $\phi'(t) > 0$ for all $t \in \mathbb{R}$. Therefore, since,

$$
\begin{aligned}
\mathcal{F}'(t-1) &= \frac{d}{dt}\log \mathcal{F}(t)\\
&= \frac{\mathcal{F}'(t)}{\mathcal{F}(t)}
\end{aligned}
$$

By infinite descent we must have $\mathcal{F}'(t) > 0$. Therefore of this nature we have a differentiable (equally as bijective) inverse $\mathcal{A} = \mathcal{F}^{-1}(t) : \mathbb{R} \to (-2, \infty)$. In laymen's terms, amongst the jargon of people who study tetration; one calls this the super-logarithm. It is a continuously differentiable Abel function of e^t. En drame,

$$
\mathcal{A}(e^t) = \mathcal{A}(t) + 1
$$

These facts will be reinforced throughout this paper. Nonetheless it helps to introduce them when they can be conveniently introduced. We state this less than drastic theorem below.

Theorem 3.1. *For some $\omega \in \mathbb{R}$ there exists a continuously differentiable tetration function $\mathcal{F}(t) : (-2, \infty) \to \mathbb{R}$ such that $\mathcal{F}'(t) > 0$ and $\mathcal{F}$ is a bijection. This function $\mathcal{F}(t)$ can be expressed as,*

$$
\mathcal{F}(t) = \lim_{n\to\infty} \log\log\cdots(n\text{ times})\cdots\log\phi(t+\omega+n)
$$

Where,

$$
\phi(t) = \Omega_{j=1}^{\infty} e^{t-j+z} \bullet z\Big|_{z=0}
$$

This provides us with a continuously differentiable function $\mathcal{F}$ defined for $(-2, \infty)$, but it sadly says nothing of the case for complex numbers. This proves to be a much more exhausting challenge. But the challenge is perfectly manageable.

To set the stage we'll work on an easier case. The function ϕ is periodic with period $2\pi i$, and therefore a very similar argument as that of above allows us to construct $\mathcal{F}(s)$ for $s \in \mathbb{R} + 2\pi i k$ for $k \in \mathbb{Z}$ when $k \neq 0$. $\mathcal{F}$ will not be real valued on these lines, but ϕ will be, which allows for the same bounds. The same initial conditions are given as $\tau_1(s) = s$ and $\tau_0(s) = 0$, so the convergence follows for $t > T$.

As that,

$$
\begin{aligned}
|\tau_{n+1}(t+2\pi ik) - \tau_n(t+2\pi ik)| &\leq \frac{|\tau_n(t+1+2\pi ik) - \tau_{n-1}(t+1+2\pi ik)|}{|\phi(t+1)|}\\
&\leq \frac{|t+2\pi ik+n|}{\prod_{j=1}^{n}|\phi(t+j)|}
\end{aligned}
$$

Therefore,

$$\mathcal{F}(t + 2\pi ik) = \phi(t + \omega) + \tau(t + \omega + 2\pi ik)$$

Is a continuously differentiable function. Going further, if $\mathcal{F}(s) = 0$ then $s = -1$ necessarily; orbits of log on complex numbers with non zero imaginary part stay away from 0. (This statement will be made much clearer in the coming sections.) Therefore, we can extend $\mathcal{F}$ to $\mathbb{R} + 2\pi ik$ for all $k \in \mathbb{Z}$ with $k \neq 0$ by taking repeated logarithms.

An important note, as $s \to -\infty$, the functions $\tau(s + 2\pi ik) \to L_k$ where L_k is a fixed point of the exponential map. The derivation of this comes from the limit $\lim_{s\to-\infty} \phi(s) = 0$ so,

$$\begin{aligned}
\lim_{s\to-\infty} e^{\phi(s)+\tau(s+2\pi ik)} &= \lim_{s\to-\infty} \phi(s+1) + \tau(s+1+2\pi ik) \\
\lim_{s\to-\infty} e^{\tau(s+2\pi ik)} &= \lim_{s\to-\infty} \tau(s+1+2\pi ik) \\
e^{\tau(-\infty+2\pi ik)} &= \tau(-\infty+2\pi ik) \\
e^{L_k} &= L_k
\end{aligned}$$

The fixed points satisfy the conjugate identity $\overline{L^k} = L^{-k}$. Once we've shown $\mathcal{F}$ is analytic; this can be derived because $\overline{\mathcal{F}(s)} = \mathcal{F}(\overline{s})$. Which follows because $\mathcal{F} : \mathbb{R}^+ \to \mathbb{R}^+$. The idea then, depending on how we limit to negative infinity, we get different fixed points. This relationship will be mirrored for $\lim_{t\to-\infty} \mathcal{F}(t + iy) = L$ for arbitrary $y \in \mathbb{R}$. In that, as $\Re(s) \to -\infty$ the function $\mathcal{F}(s) \to L$ (or we hit a branch-cut along the way) where L is a fixed point of the exponential map. This can be better codified if one thinks: iterated logarithms converge to a repelling fixed point of e^s or we get a branching problem (I.e: F hits zero somewhere).

Of the following discussion we can see that $\mathcal{F}(s)$ is continuously differentiable on the lines $\mathbb{R} + 2\pi ik$ for $k \neq 0$ and continuously differentiable on $(-2, \infty)$. We were able to obtain this result because $\tau_{n+1} - \tau_n$ was bounded by a product of the form $\dfrac{|t + n + 2\pi ik|}{\prod_{k=1}^{n} \phi(t + k)}$. Since $\phi(t)$ grows super-exponentially, this product is very well behaved. We had yet to mention this fact but $\dfrac{1}{\phi(t)}$ is smaller than any iterate of the exponential as t grows. Which is $\frac{1}{\phi(t)} \leq \frac{1}{\exp^{\circ n}(t)}$ for large enough $t \geq T$.

In order to get τ to be holomorphic it would be required that we can control this product well in the complex plane. Doing this is a bit of a hassle. It necessitates us understanding how ϕ grows. The main hurdle of this paper comes in what follows.

4 Obtaining lower bounds on ϕ

This section will focus on the function $\phi(s)$ in the strip $0 \leq \Im(s) = y \leq \pi$ for large positive values of $t = \Re(s)$. The idea is to show for some large value T the function $|\phi(t+iy)| > 1+\epsilon$ for some $\epsilon > 0$ and all $t > T$. Despite: the simplicity of this statement; the fact ϕ grows super-exponentially on the real line; this statement proves to be the most difficult part of this construction to digest.

Commence by assuming that $|\phi(t+iy)| < M \in \mathbb{R}^+$ for all $t \geq T$ and y fixed. Then by the functional equation,

$$|\phi(t+1+iy)| = e^{t+\Re\phi(t+iy)} \geq e^{t-M}$$

But this equation tends to infinity as $t \to \infty$. Therefore, we know instantly that $\phi(t+iy)$ is unbounded as $t \to \infty$. The more important question is whether it grows towards infinity or oscillates back and forth.

As some preliminary remarks; this result is highly non-trivial. It meshes well with intuition, but requires a good amount of heavy lifting. We'll need to reduce the case for all $0 \leq y \leq \pi$ to the case $y = \pi$; which in essence is the worst our function can grow like. The author had a more ambitious idea for this section, but kept on failing; and upon failure gathered what he could to prove that eventually $\phi(t+i\pi)$ is greater than $1+\epsilon$ and stays greater. The ambitious proof the author, at first, thought was self-apparent, was that $\phi(t+\pi i)$ grows asymptotic to $\log^{\circ\epsilon}(t)$ for $\epsilon \to 0$ as $t \to \infty$. And that which he cannot prove for the life of him. But luckily, we can at least prove it stays above a point. And that's really all that's needed–fortunately.

In short; ϕ does not grow super-exponentially in the complex plane as it does on the real positive line. In fact, if $|\phi(t+\pi i)| > t$ then $|\phi(t+1+\pi i)| = e^{t-|\phi(t+\pi i)|} \leq 1$. As such $\phi(t+\pi i)$ must be $o(t)$.

The only fact the author can think of to explain why this is so relies on a heavy study of complex dynamics. But, if $\mathcal{N}$ is a neighborhood in $\mathbb{C}$ then the orbits $\exp^{\circ k}(\mathcal{N})$ are dense in $\mathbb{C}$ [5]. Therefore, the neighborhoods $\mathcal{F}(\mathcal{N}+k)$ must be dense in $\mathbb{C}$. And to do this, it must grow relaxed in some parts. In accordance, ϕ must grow slow in some parts; or otherwise the entire construction will collapse.

The author couldn't derive an exact asymptotic for ϕ, but he guesses the main term is something just less than t; he hopes something like $\log^{\circ 1/t}(t)$; which can be expressed as $\mathcal{F}(\mathcal{A}(t)-1/t)$.

* * *

To better accustom ourselves to the behaviour of ϕ in the complex plane we can rewrite our function. It'll look a bit stranger in this form but,

$$\psi_y(t) = e^{-iy}\phi(t+iy) = \Omega_{j=1}^{\infty} e^{t-j+e^{iy}z} \bullet z\Big|_{z=0}$$

Then,

$$\psi_y(t) = e^{t - 1 + e^{iy} e^{t - 2 + e^{iy} e^{\cdots}}}$$

By analyzing the partial compositions,

$$\psi_n(t) = \underset{j=1}{\overset{n}{\Omega}} e^{t - j + e^{iy} z} \bullet z \Big|_{z=0}$$

They satisfy the identity,

$$\psi_{n+1}(t+1) = e^{t + e^{iy}\psi_n(t)}$$

Therefore,

$$\psi_y(t+1) = e^{t + e^{iy}\psi_y(t)}$$

And we can obtain the crude estimate,

$$|\psi_y(t+1)| \geq e^{t - |\psi_y(t)|}$$

Now this estimate gets its worse when $y = \pi$ where,

$$\psi_\pi(t+1) = e^{t - \psi_\pi(t)}$$

So, the heuristic is, if we could construct a lower bound for this case we could bound from below the other cases. So the manner of proof will involve working firstly on the case where $y = \pi$. As a preliminary remark, if the reader hasn't noticed: When $y = \pi$ the function $\psi_\pi(t)$ is real valued and strictly positive.

To this end,

$$\psi(t) = \psi_\pi(t) = \underset{j=1}{\overset{\infty}{\Omega}} e^{t-j-z} \bullet z \Big|_{z=0}$$

Where we notice that all we've really done is swap the sign in the z term from our definition of $\phi(t)$. A subtle difference, which as Dorothy would put it–I don't think we're in Kansas anymore, Toto. It is hereupon we have to fiddle with our notation a bit. We will write,

$$\psi_m(t,z) = \underset{j=1}{\overset{2m+1}{\Omega}} e^{t-j-z} \bullet z$$

The reader should note the $2m+1$ in the upper index of our composition. And they should note that $\psi_m(t,0)$ converges uniformly to a continuously differentiable function as $m \to \infty$. Where now our recursion is a tad different but,

$$\psi_{m+1}(t,z) = \psi_m(t, e^{t - 2m - 1 - e^{t-2m-2-z}})$$

We've doubled up on the exponentials here because we want to use that,

$$e^{t+1-e^{t-z}}$$

Starts to look like $e^{-\delta e^t}$ for some $\delta > 0$; and this has very fast convergence to zero $t \to \infty$. But additionally,

$$e^{t-2m-1-e^{t-2m-2-z}}$$

Tends to zero like $\mathcal{O}(e^{-m})$ as $m \to \infty$ as well. We've chosen the upper index $2m+1$ because each of these $\psi_m(t)$ grow exponentially. This leads us to a more regular idea of what our infinite compositions behave like; which is,

$$\psi_m(t,z) \to \psi(t) = \lim_{m\to\infty} \psi_m(t,0)$$

Which means, regardless of what z-value we choose the result still converges to the same limit as when we set $z = 0$. We can think of this as iterations converging to a fixed point locally in z. This is perhaps the best way to think of it. So, as per the above,

Theorem 4.1. *The function $\phi(s)$ can be represented as,*

$$\phi(s) = \mathop{\Omega}_{j=1}^{\infty} e^{s-j+z} \bullet z$$

for all $z \in \mathbb{C}$.

Proof. The limit function,

$$\begin{aligned}
\phi(s,z) &= \mathop{\Omega}_{j=1}^{\infty} e^{s-j+z} \bullet z \\
&= \mathop{\Omega}_{j=1}^{m} e^{s-j+z} \bullet \mathop{\Omega}_{j=m+1}^{\infty} e^{s-j+z} \bullet z \\
&= \mathop{\Omega}_{j=1}^{m} e^{s-j+z} \bullet \epsilon \bullet z \text{ per Lemma 2.1} \\
&\to \mathop{\Omega}_{j=1}^{\infty} e^{s-j+z} \bullet z\Big|_{z=0} \text{ as } m \to \infty
\end{aligned}$$

Because as $m \to \infty$ we get $\epsilon \to 0$. □

Now, for some $m > M$ we can take $|z| < 1$ and $t > T$ such that $\inf_{|z|<1} |\psi_m(t,z)| > 1 + \epsilon$. From this,

$$|\psi_{m+1}(t,z)| = |\psi_m(t, e^{t-2m-1-e^{t-2m-2-z}})| > 1 + \epsilon$$

Because, $||e^{t-2m-1-e^{t-2m-2-z}}||_{|z|<1} \leq \delta \leq 1$ for large enough $m > M$ and $t > T$. Therefore, by induction the result follows.

Theorem 4.2. *Let,*

$$\psi(t) = \Omega_{j=1}^{\infty} e^{t-j-z} \bullet z$$

Then for some $\epsilon > 0$ there exists some $T > 0$ such that, for all $t > T$:

$$\psi(t) \geq 1 + \epsilon$$

To derive the result for the general case, we need only run a cranking mechanism. The function,

$$\frac{d}{dy}|\psi_y(t)| = 0 \text{ iff } y = k\pi \text{ for } k \in \mathbb{Z}$$

Where when k is even corresponds to maxima; and when k is odd corresponds to minima. Since this function is periodic in y with period 2π we know instantaneously that $|\psi_y(t)| \geq \psi_\pi(t)$. We leave the cranking to the reader; it's fairly laborious to say the least; but doesn't take much more than punching in the values.

From henceforth, we'll write $\lambda = \frac{1}{1+\epsilon} < 1$. And we can see that,

$$\frac{1}{|\phi(t+iy)|} \leq \lambda < 1$$

5 Showing τ is holomorphic

To show τ is holomorphic only requires a few careful observations. Firstly, there exists some $T \in \mathbb{R}^+$ such that for all $t = \Re(s) \geq T$ the value $|\phi(s)| \geq 1 + \epsilon$. Therefore, of this form,

$$\frac{1}{\prod_{k=1}^{n} |\phi(s+k)|} \leq \lambda^n$$

The nature of which is that this product geometrically converges to 0. It is not quite super-exponential–like what happens on the real-line, but nonetheless we've done fairly well for ourselves. This is a very convenient bound. When we look at our τ functions,

$$\begin{aligned} |\tau_{n+1}(s) - \tau_n(s)| &\leq \frac{1}{|\phi(s+1)|}|\tau_n(s+1) - \tau_{n-1}(s+1)| \\ &\leq \lambda|\tau_n(s+1) - \tau_{n-1}(s+1)| \end{aligned}$$

Because $|\log(1+z_1) - \log(1+z_2)| \leq |z_1 - z_2|$ so long as $|z_1|, |z_2| \geq 2$; which because τ_n looks like s, and s is very large; we are all good. Remembering that $\tau_0(s) = 0$ and $\tau_1(s) = s$,

$$|\tau_{n+1}(s) - \tau_n(s)| \leq \lambda^n |s+n|$$

This converges uniformly in the strip $T \leq t = \Re(s) \leq T'$ and $y = |\Im(s)| \leq Y$ for $T, T', Y > 0$ large enough. This implies $\tau(s)$ is holomorphic. The next step is to extend τ's domain of holomorphy, from these boxes to a maximal domain. Using the functional equation, since,

$$\phi(s+1) + \tau(s+1) = e^{\phi(s) + \tau(s)}$$

By taking logarithms τ can be extended. The only difficulty we could have is if $\tau(s+1) = -\phi(s+1)$; wherein no logarithm can be taken and τ must have a singularity. We will defer a proof that this is impossible; the author doesn't know. But nonetheless, if we continue this implicit definition, the worst that happens when extending $\mathcal{F}$ is that we hit a branch cut at some point. So the nice way to say this is that,

$$\mathcal{F}(s) \text{ is holomorphic on } \mathbb{C} \text{ upto a nowhere dense set}$$

Which is really just a nice way of saying–I have no idea where the branch cuts are, or if there are even branch cuts; I'm just playing it safe. Don't sue me.

With this we state our titular theorem:

Theorem 5.1 (The Tetration Existence Theorem)**.** *The function $\mathcal{F}$ from Theorem 3.1 can be analytically continued to a function which satisfies the following properties:*

1. *$\mathcal{F}$ is holomorphic for $s \in \mathbb{C}/\mathcal{L}$ where $\mathcal{L}$ is a nowhere dense set.*
2. $\mathcal{F}(s+1) = e^{\mathcal{F}(s)}$
3. $\mathcal{F}(0) = 1$
4. $\mathcal{F} : \mathbb{R}^+ \to \mathbb{R}^+$, *and* $\mathcal{F}' : \mathbb{R}^+ \to \mathbb{R}^+$

It is best described; for some $\omega \in \mathbb{R}$; as,

$$\lim_{n\to\infty} \log\log ...(n \text{ times}).... \log \phi(s + \omega + n) = \mathcal{F}(s) = e \uparrow\uparrow s$$

In Conclusion

We have constructed a function, but there lies a more daunting problem. It is necessary to describe this function more acutely. The author is unsure of the behaviour $e \uparrow\uparrow t + iy$ as $y \to \infty$, nor of which fixed points of e^s the function $e \uparrow\uparrow t + iy$ tends to as $t \to -\infty$ depending on y. Just as well, he knows no method of constructing a uniqueness criterion for this function depending entirely on its behaviour on the real positive line. Though, he suspects the *eventual* monotonicity of each derivative may suffice–perhaps hinging on some extraneous growth condition. We haven't proved this here; the author considers it an insufficient theorem, and would much rather have monotonicity on all of $\mathbb{R}^+$ for all $\mathcal{F}^{(n)}$–not just eventually for large enough $t > T$. *Can the reader prove this: For all n there exists T such for $t > T$ the function $\mathcal{F}^{(n)}(t) > 0$? If they can, can they prove the stronger result that $F^{(n)}(t) > 0$ for all $t \in \mathbb{R}^+$? The author would be eternally grateful.*

We do not know much about this solution qualitatively. Does this solution agree with Kneser's solution? Does this solution agree with any other solution? Where are the zeroes of $e \uparrow\uparrow s$? And consequently, where are the branch cuts? The author doesn't; but he hopes to find out; and hopes others do as well.

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