Analytic Tetration

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In this short note, we propose a generalization of the tetration to fractional and complex heights given by an explicit series representation.

We denote tetration as ${}^{n}a$, where a is called the base and n is called the height. For $n \in \mathbb{N}$ the tetration is defined by the recurrence relation:

$${}^{0}a = 1, \quad {}^{(n+1)}a = a^{\left({}^{n}a\right)}.$$

We restrict our attention to bases in the range $e^{-e} \leq a \leq e^{1/e}$, where the following limit exists:

$${}^{\infty}a = \lim_{n\to\infty} {}^{n}a = \exp\left(-W\left(-\ln a\right)\right),$$

where $W(z)$ denotes the Lambert W-function.

Our goal is to define an analytic function of a complex variable z that satisfies the same functional relation for all z in its domain of analyticity:

$${}^{(z+1)}a = a^{\left({}^{z}a\right)}.$$

The following series gives the desired solution:

$${}^{z}a = \lim_{m\to\infty} \sum_{n=0}^{m} \rho^{n\,(z-m)} \left(\frac{\left({}^{\infty}a\right)(-1)^{n}}{\rho^{\frac{n\,(n-1)}{2}}} + \sum_{k=0}^{n-1} \frac{\left({}^{k+m}a\right)(-1)^{k}\,\rho^{\frac{k\,(k+1)}{2} - n\,k}}{\prod_{s=1}^{k}(1-\rho^{s}) \cdot \prod_{s=1}^{n-k-1}(1-\rho^{s})} \right),$$

where $\rho = {}^{\infty}a \cdot \ln a$. This can be shown by substituting this expression into the functional relation above and expanding both sides in series in powers of ρ^{z}. We note that it is possible to get rid of the limit in this expression by replacing it with an infinite telescoping series, where all terms contain only elementary functions, the Lambert W-function, tetration with a positive integer height, and finite sums and products.

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